Binary Relations

Part One

Outline for Today

- ***Binary Relations***
  - Reasoning about connections between *pairs of* objects
- ***Equivalence Relations***
  - Reasoning about clusters

Relationships

- In CS103, you've seen examples of relationships
  - between sets:

$$A \subseteq B$$

  - between numbers:

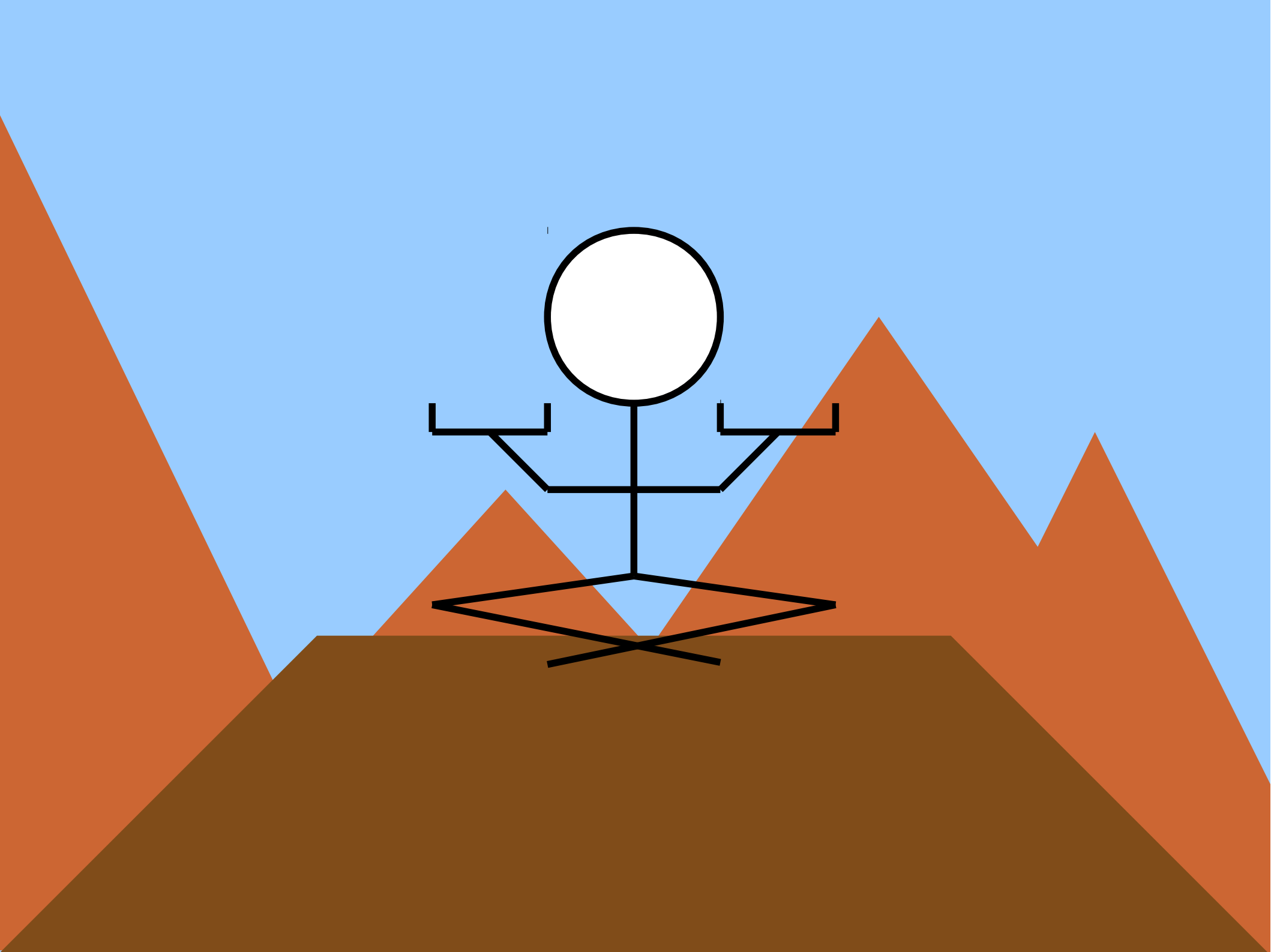
$$x < y \qquad x \equiv_k y \qquad x \leq y$$

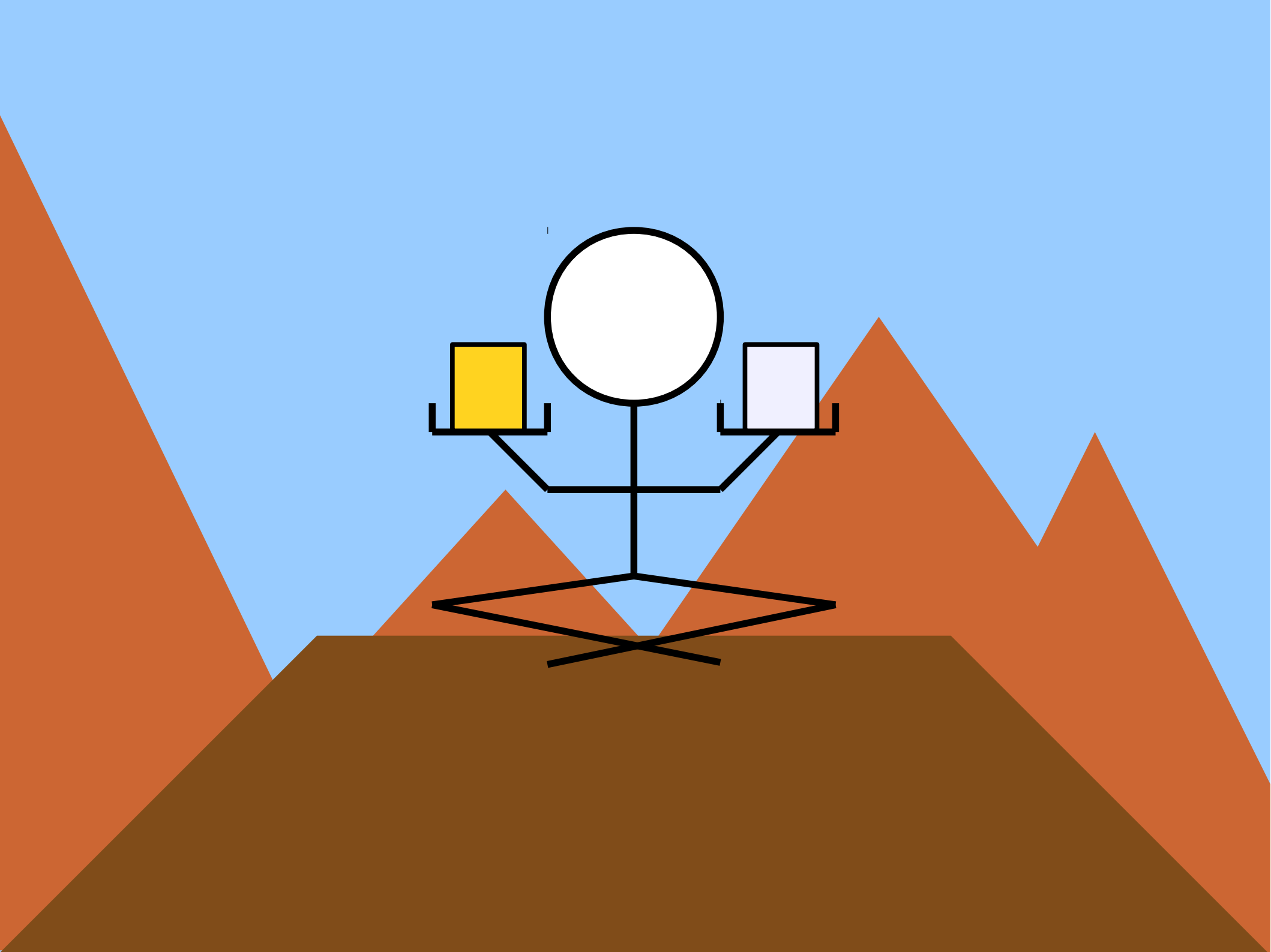
  - between people:

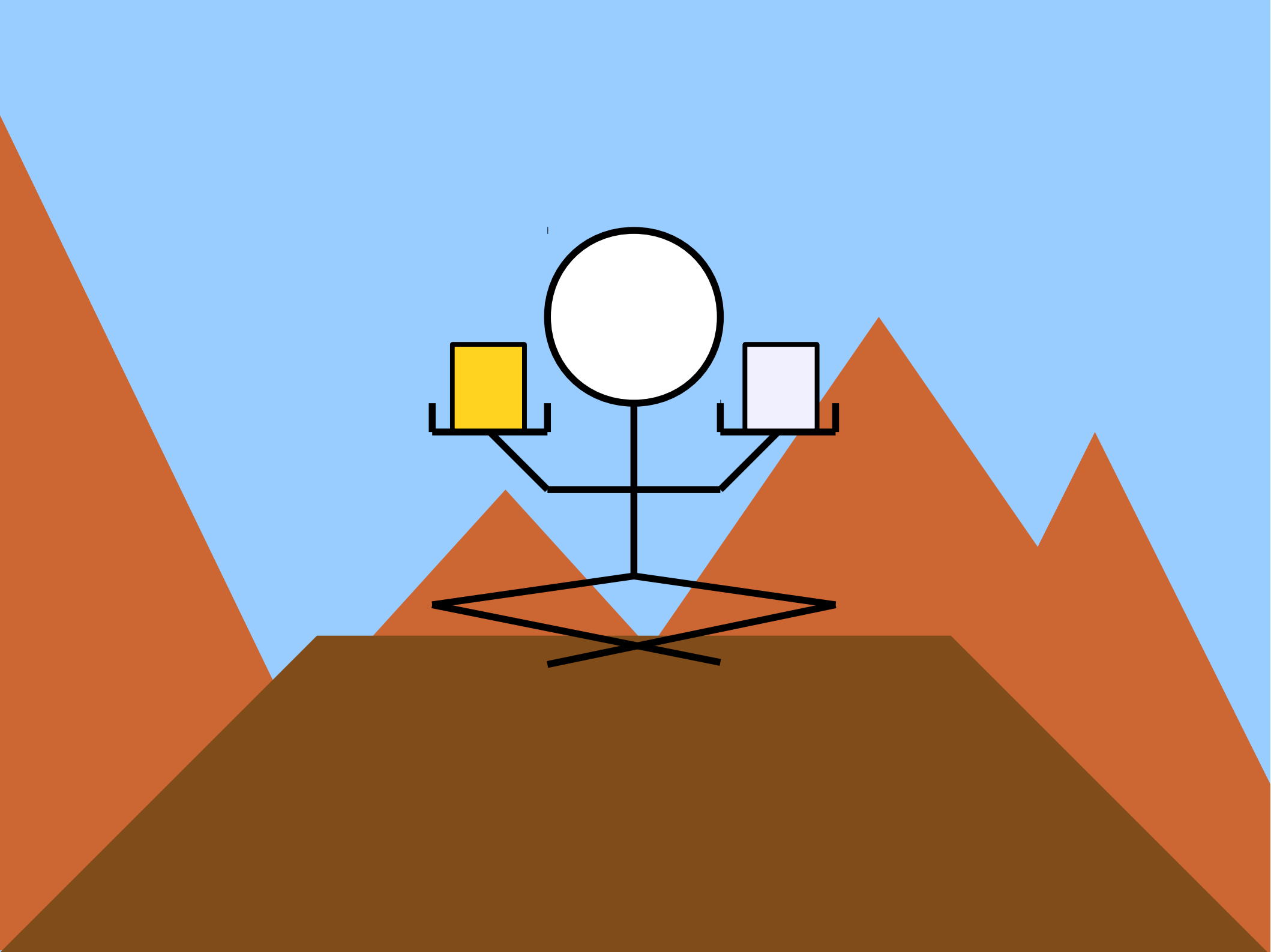
$$p \text{ loves } q$$

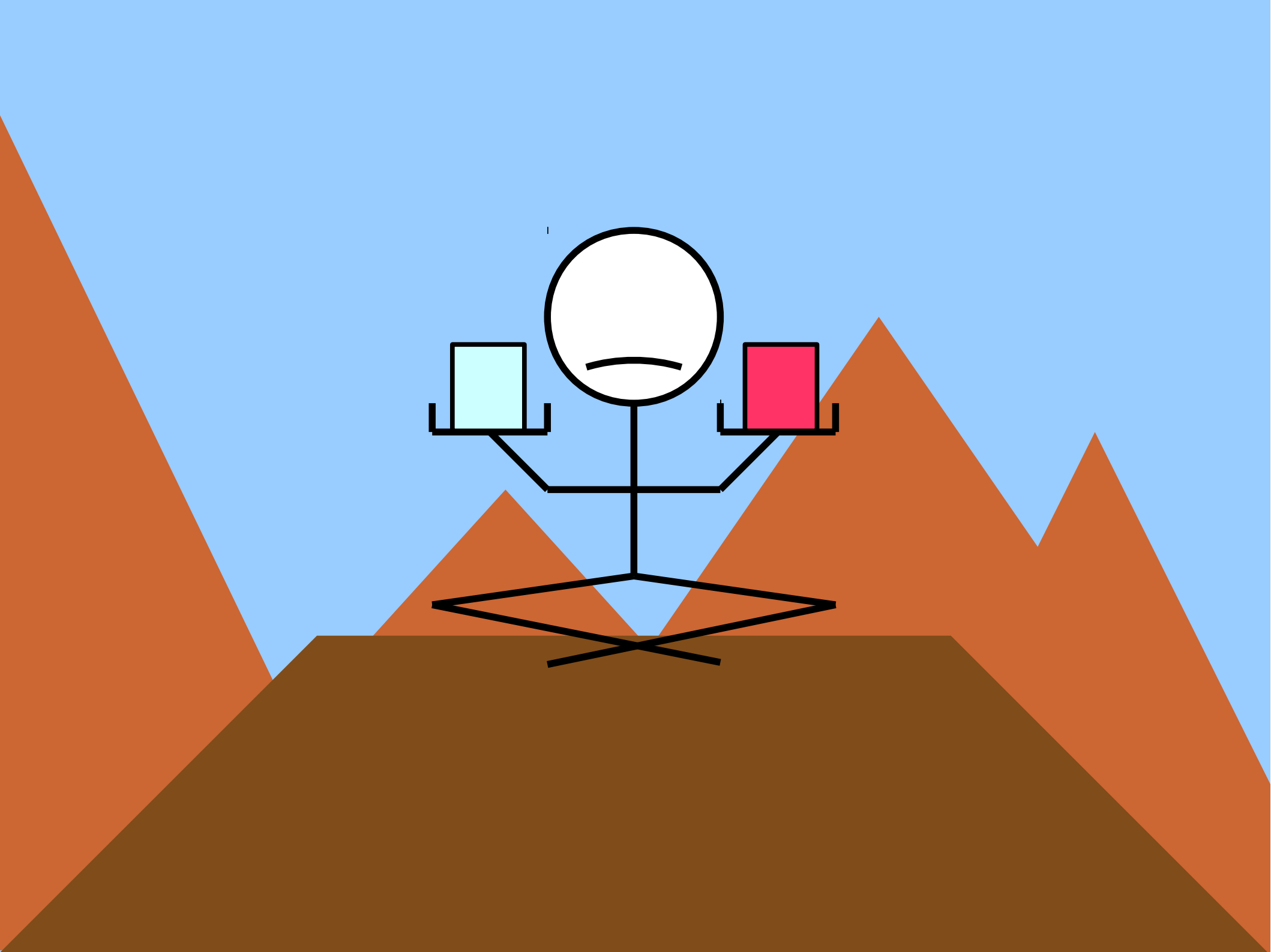
- Since these relations focus on connections *between two objects*, they are called ***binary relations***.
  - The “binary” here means “pertaining to two things,” not like computer code in 0s and 1s

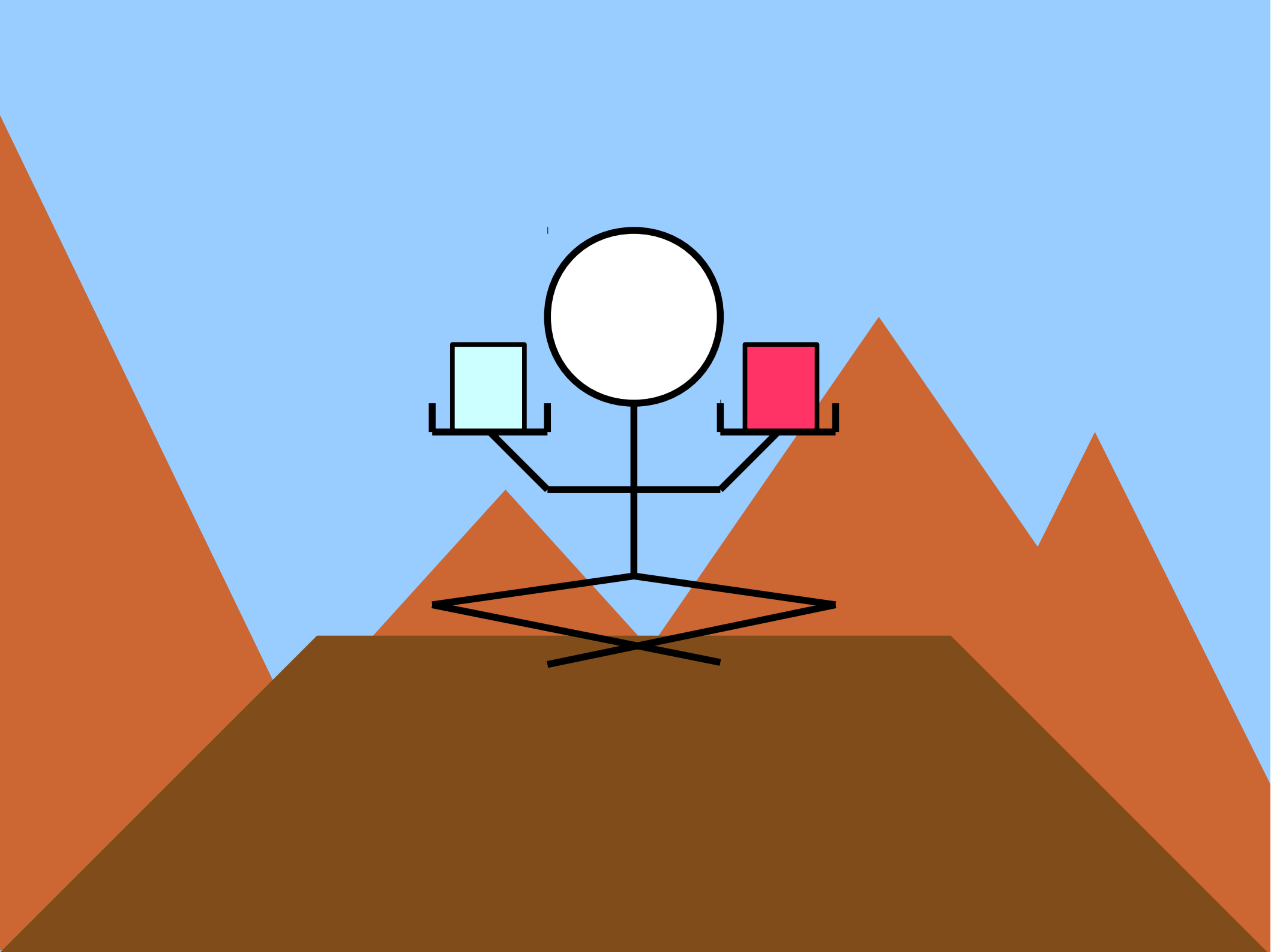
What exactly is a binary relation?

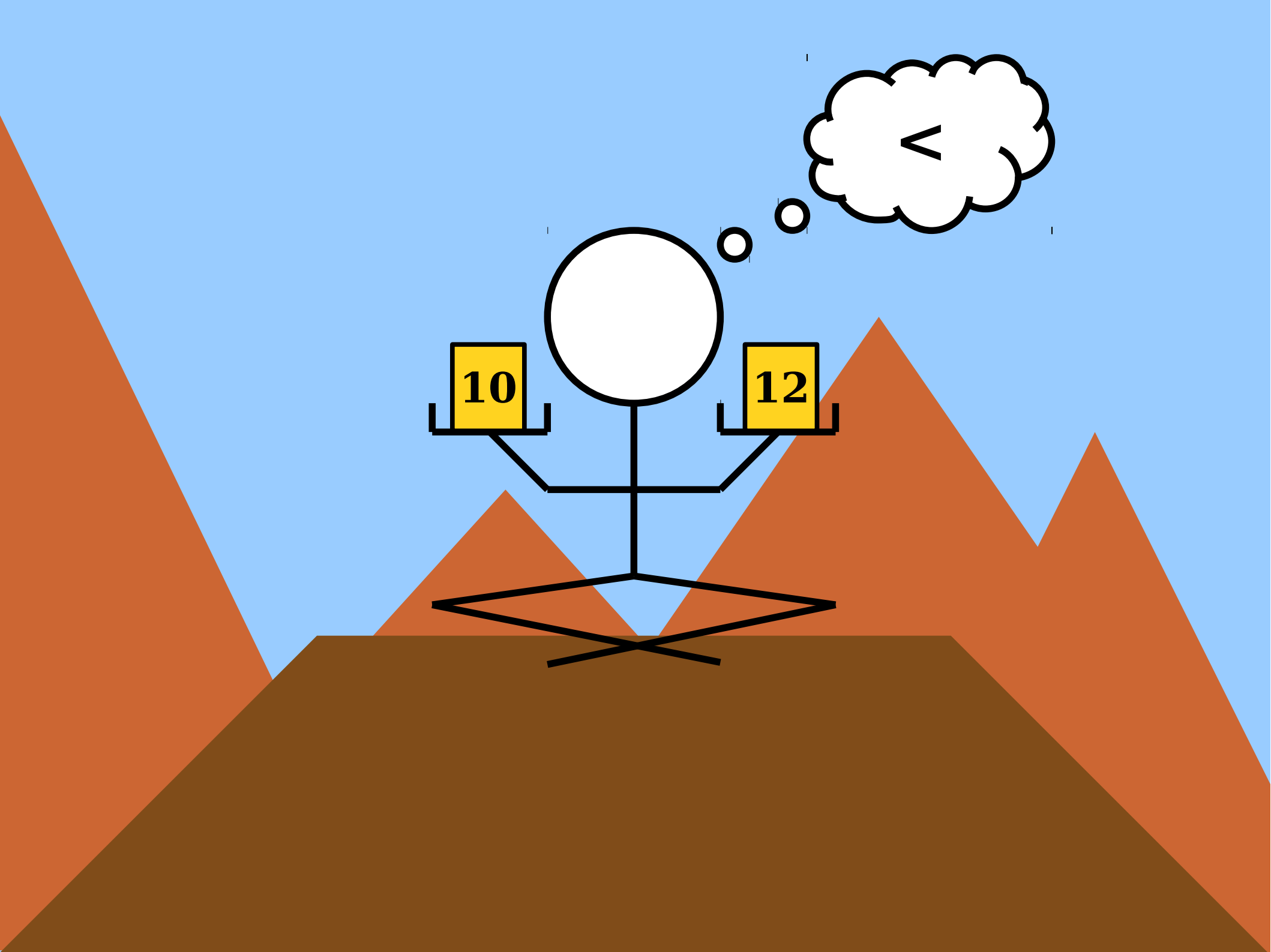








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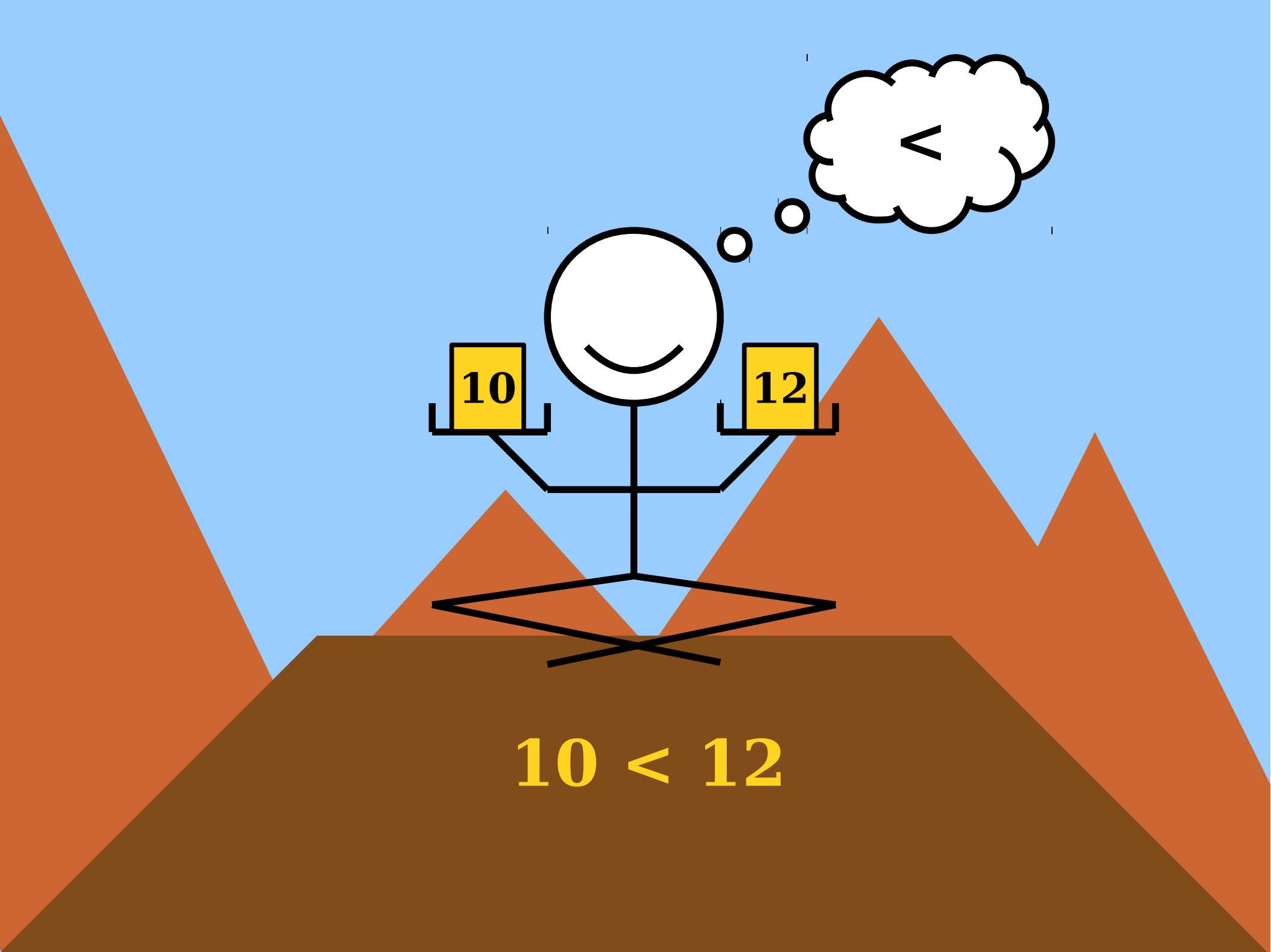
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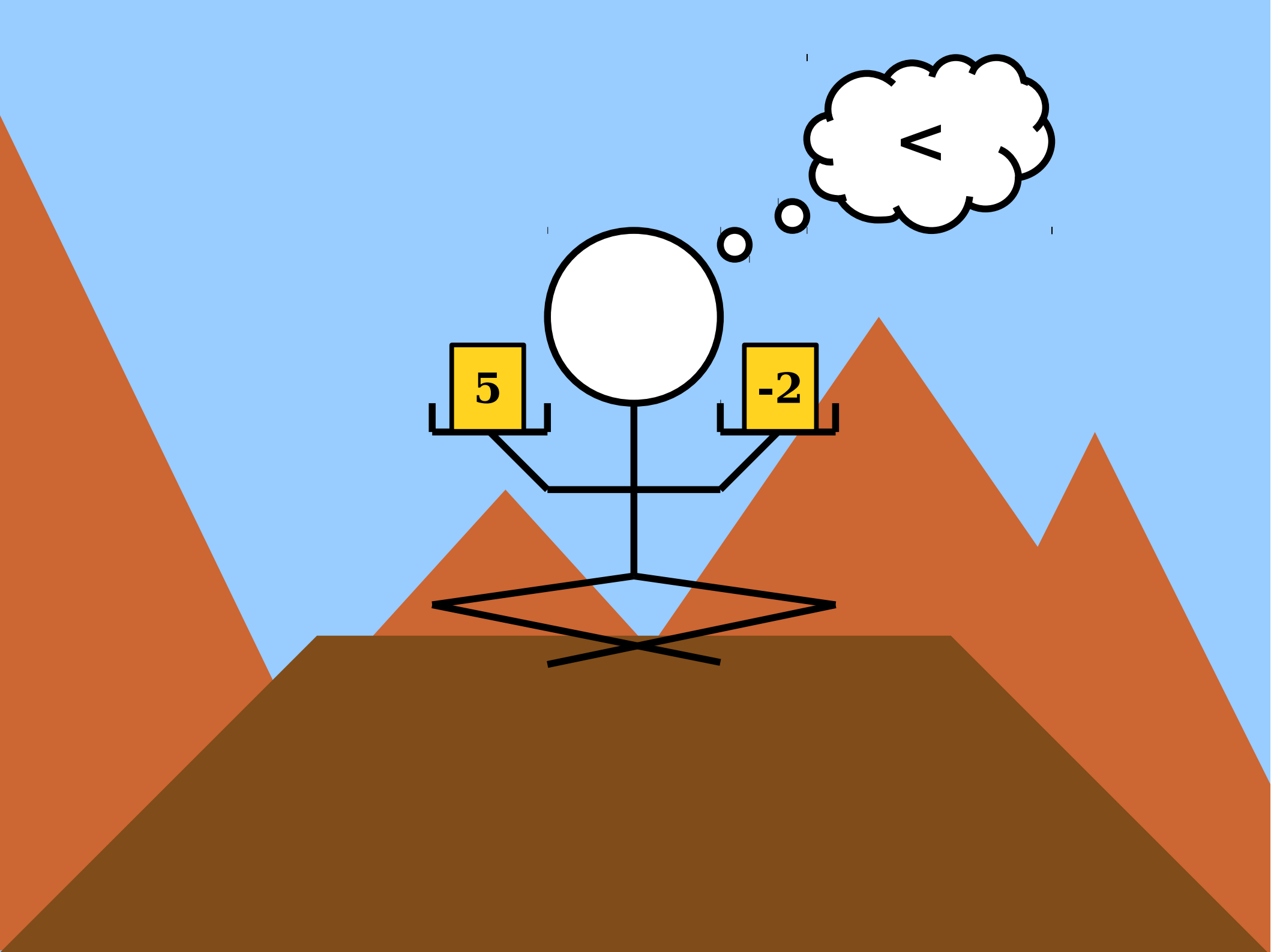
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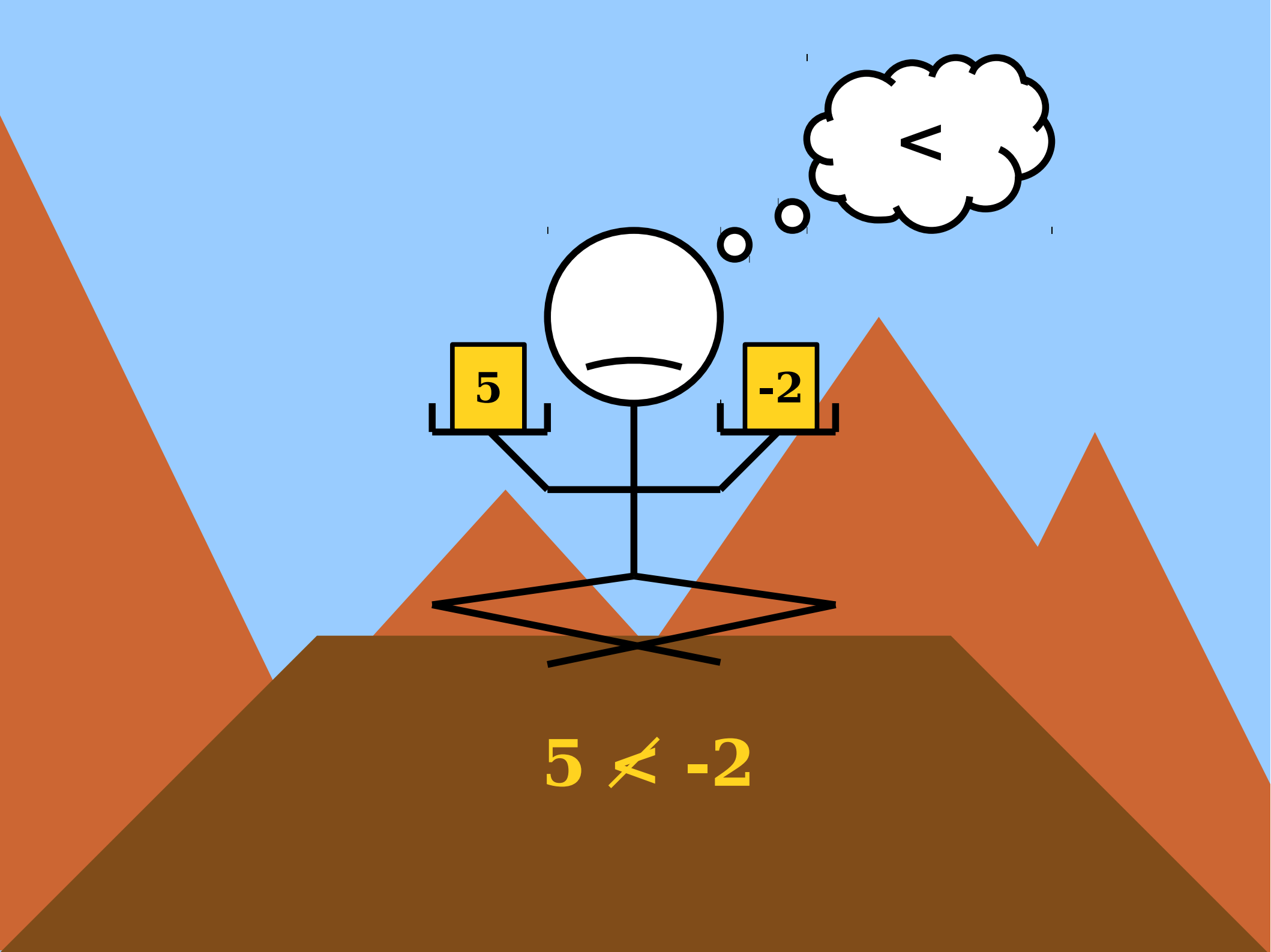
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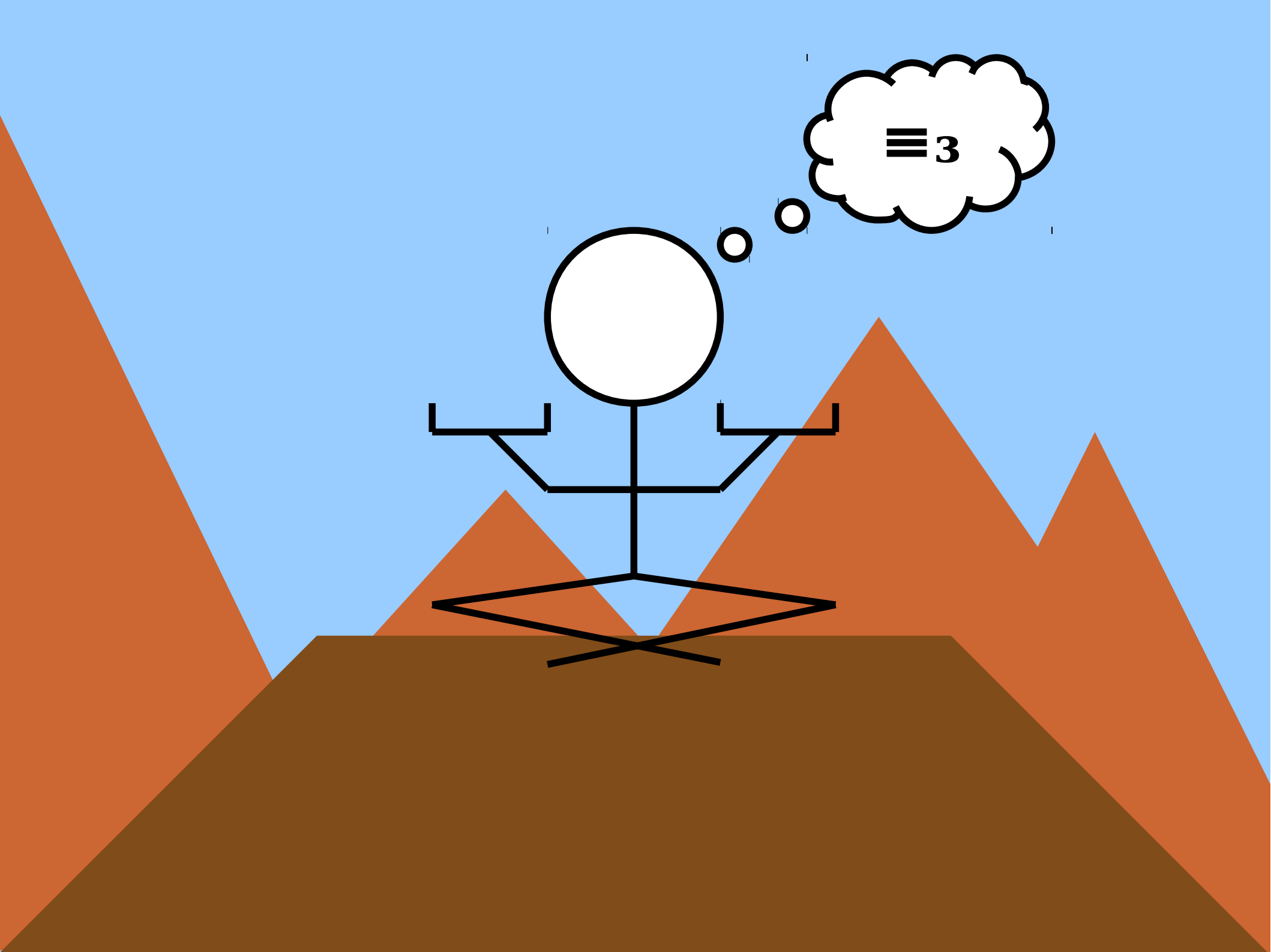
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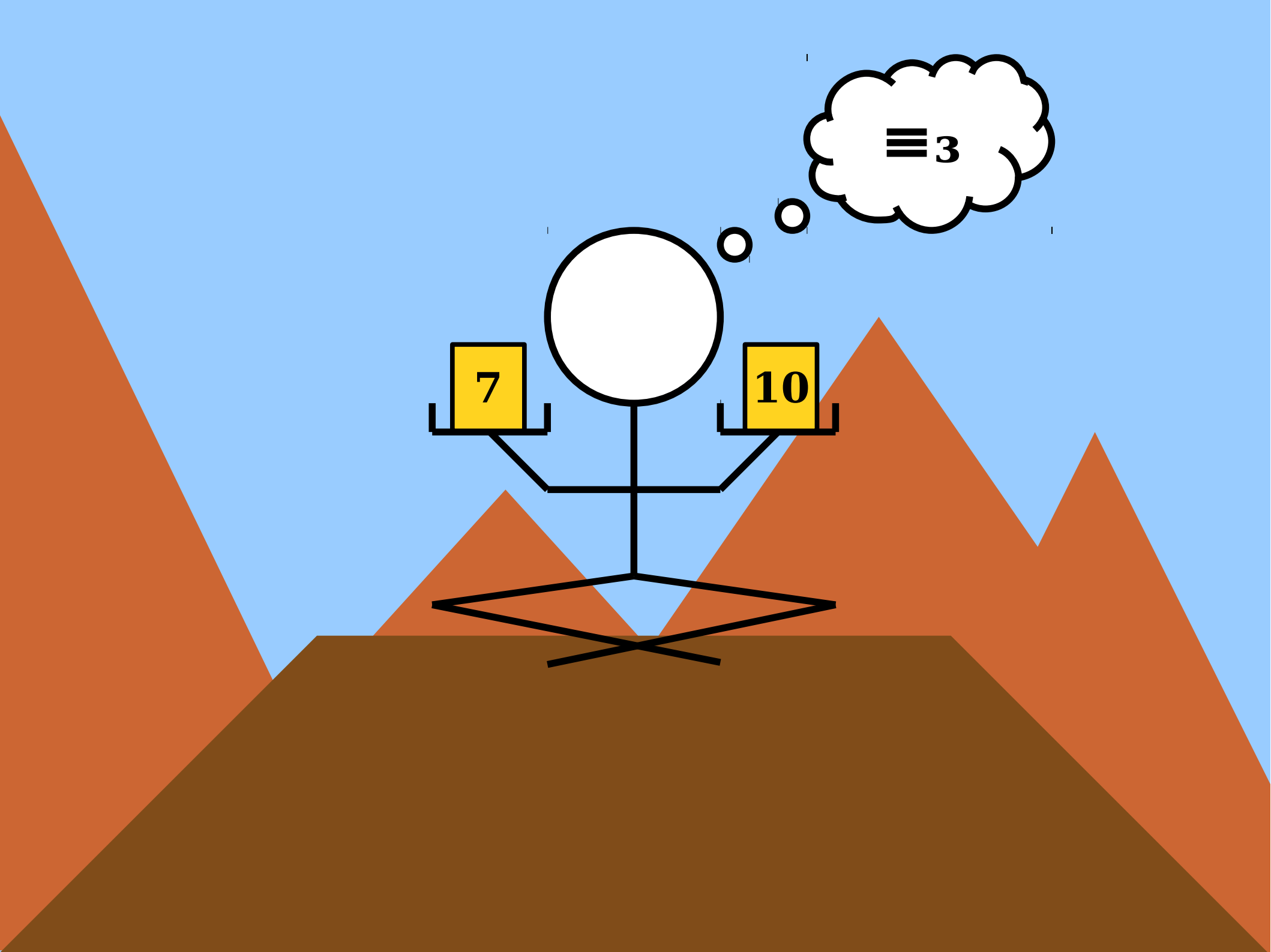
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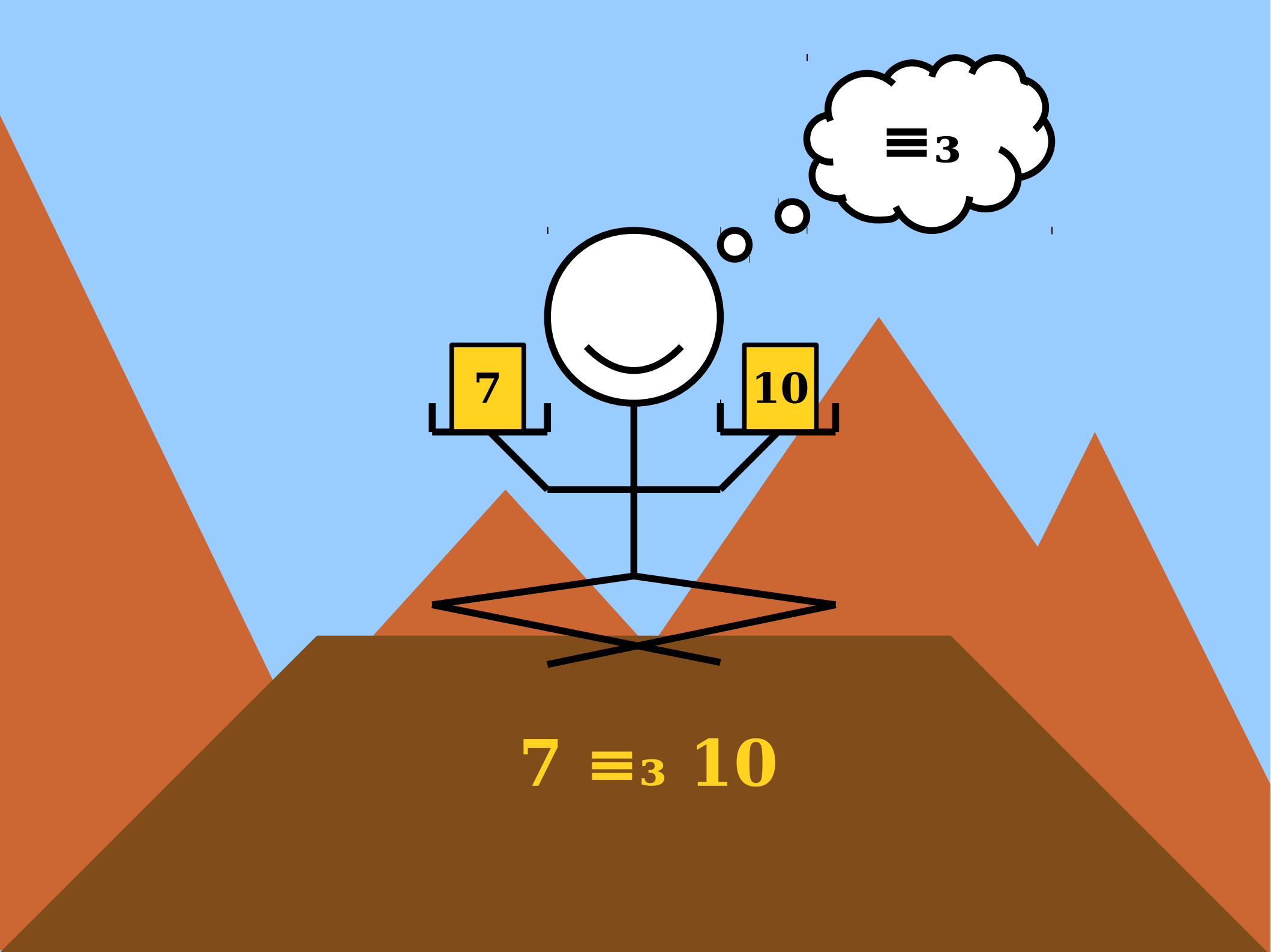
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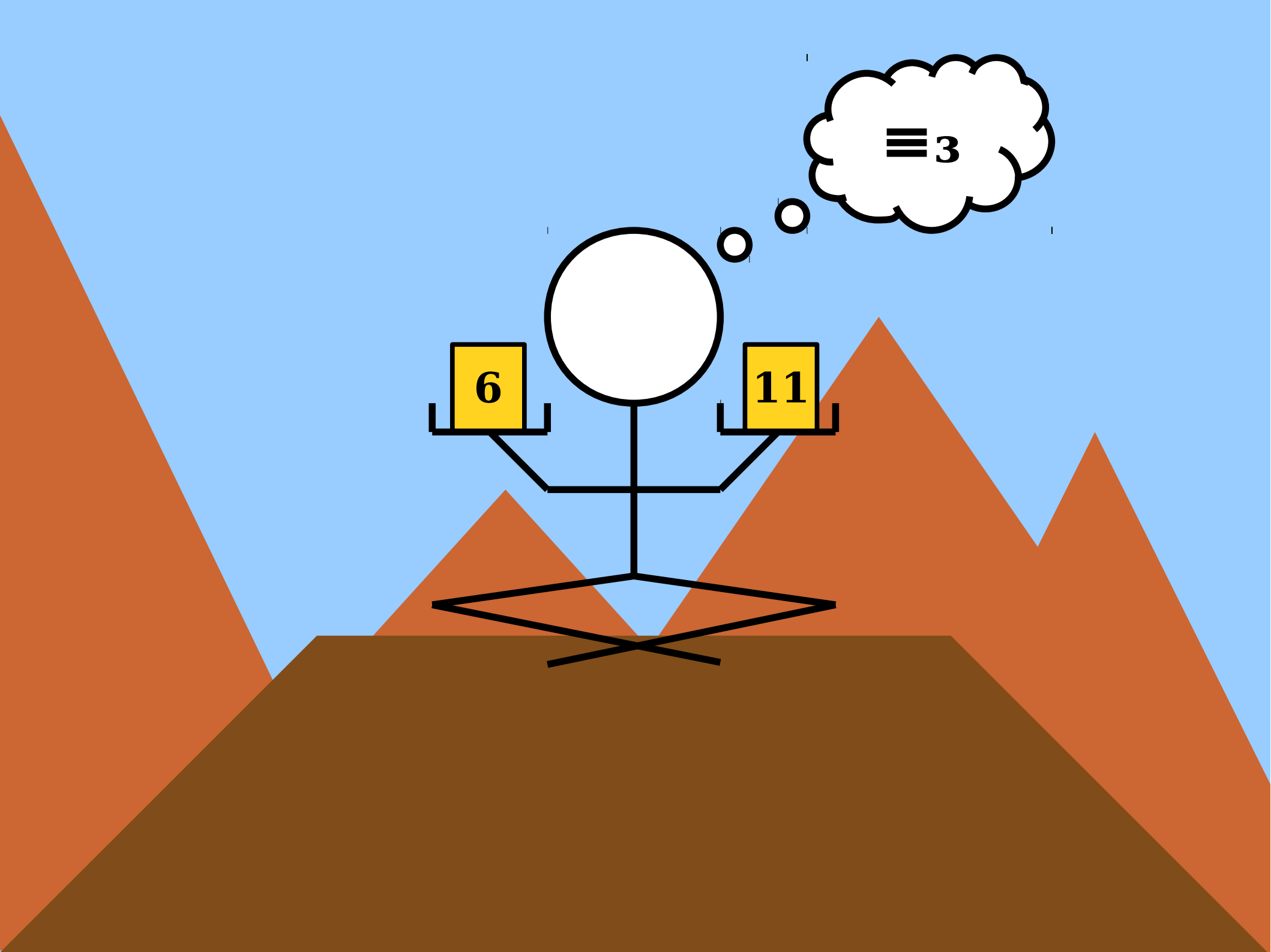
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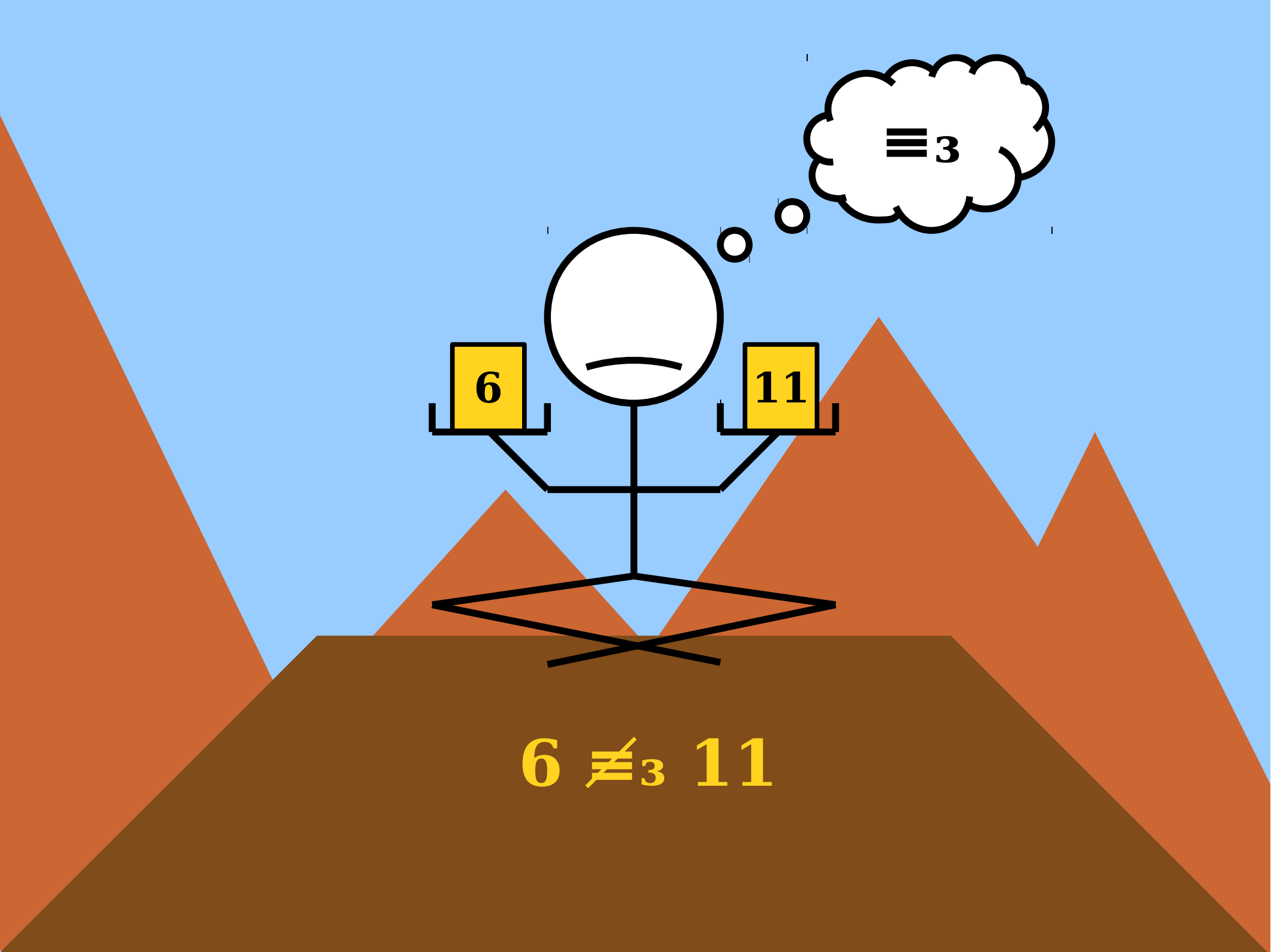
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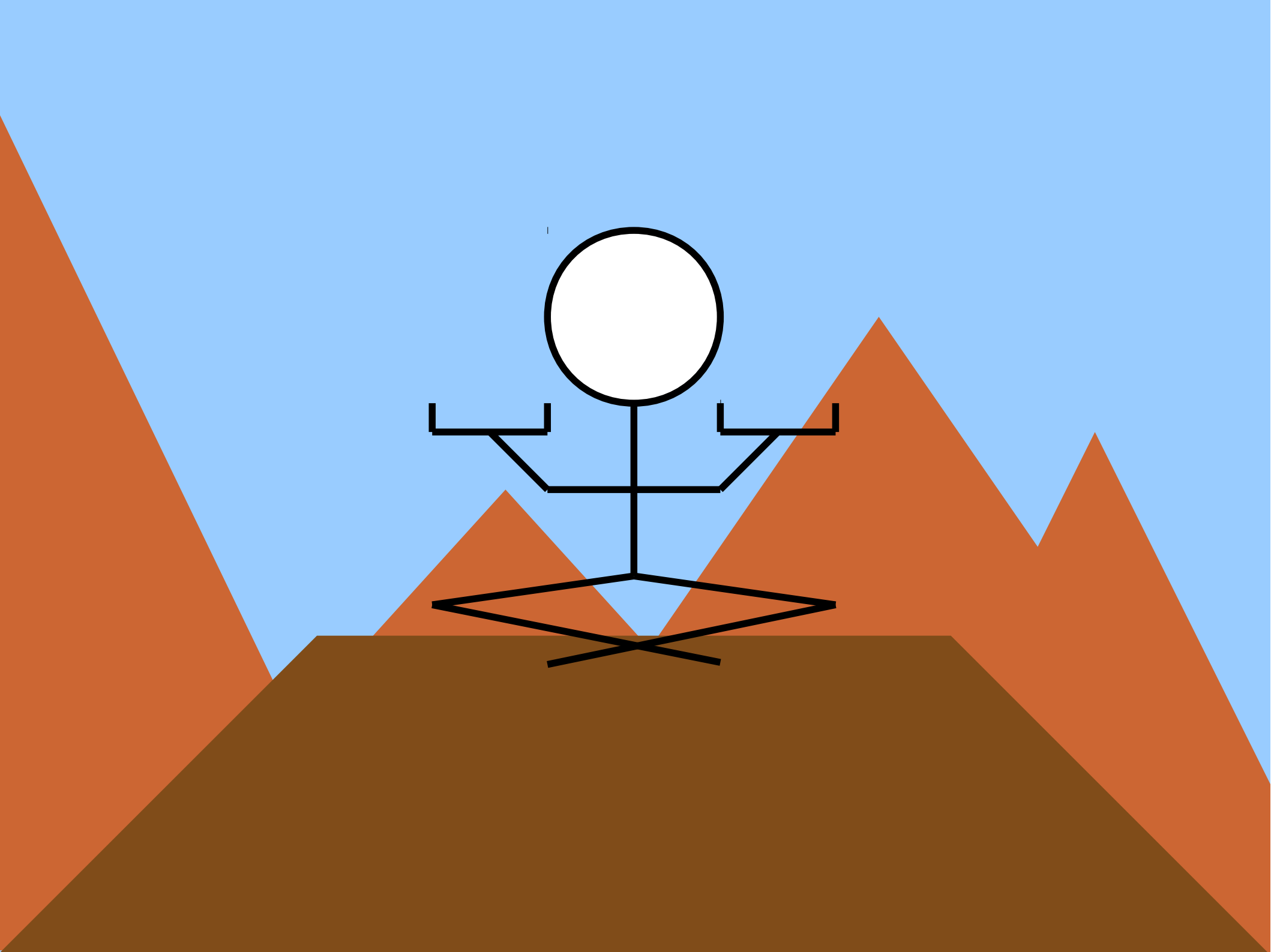

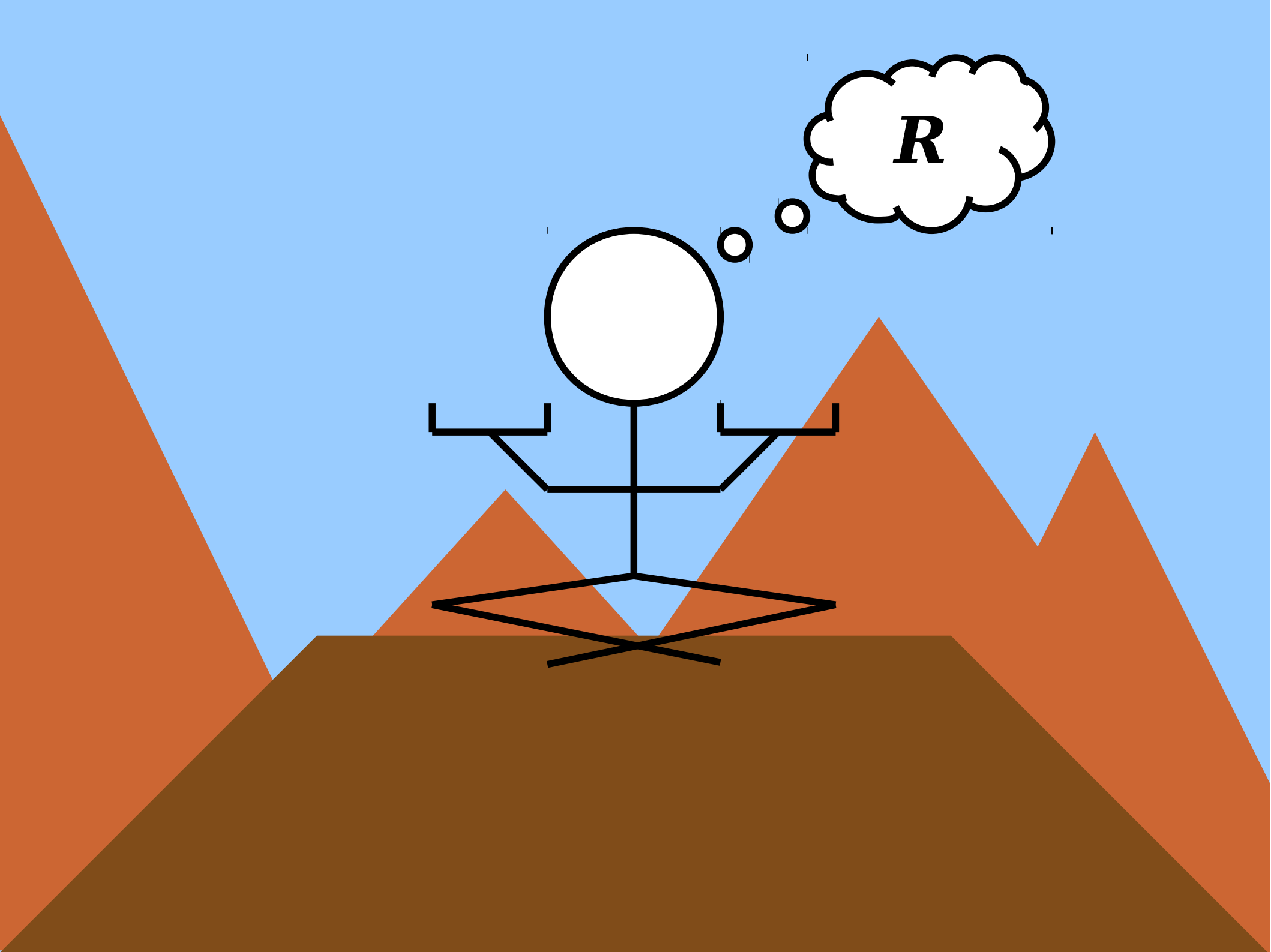
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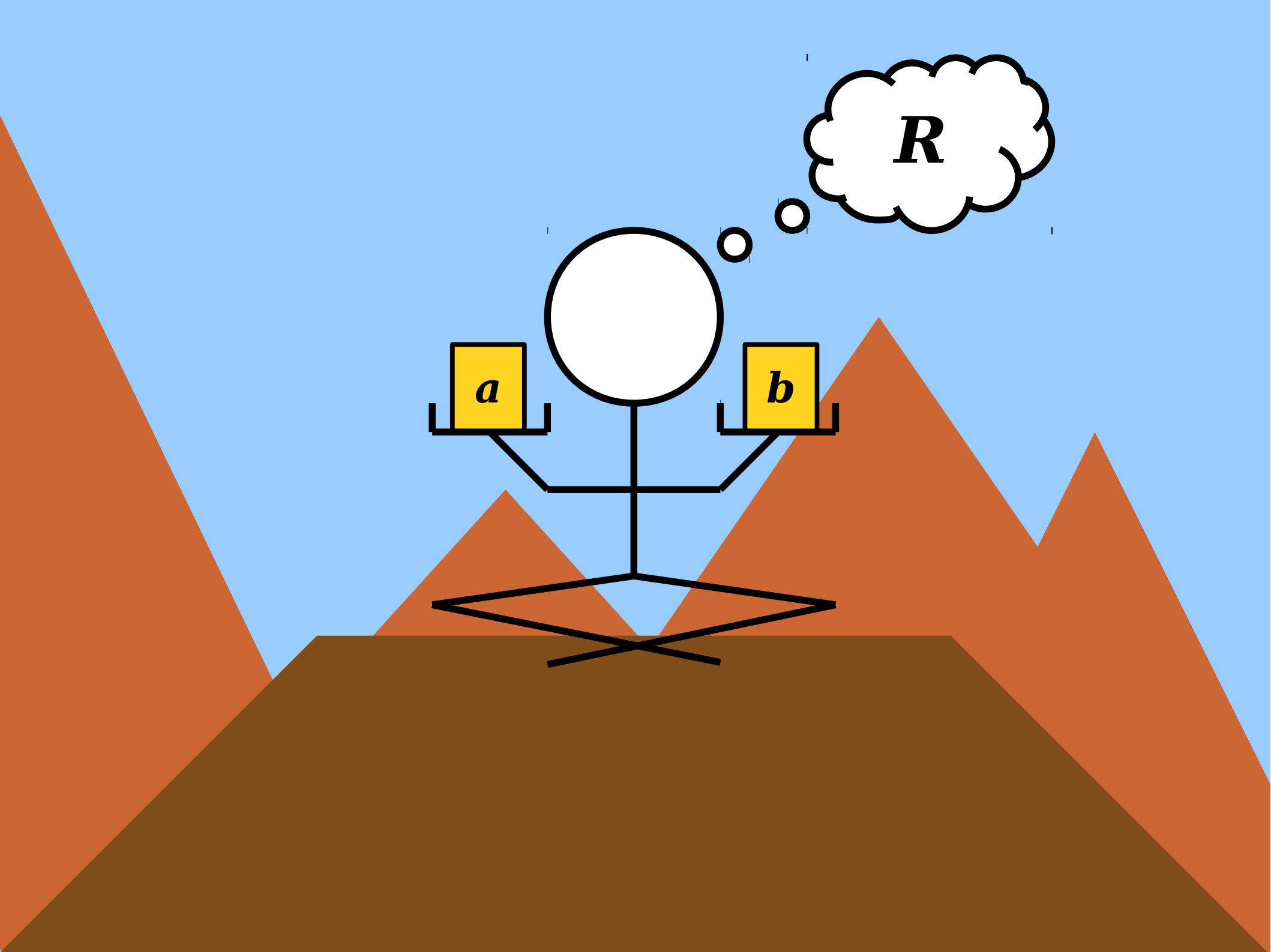
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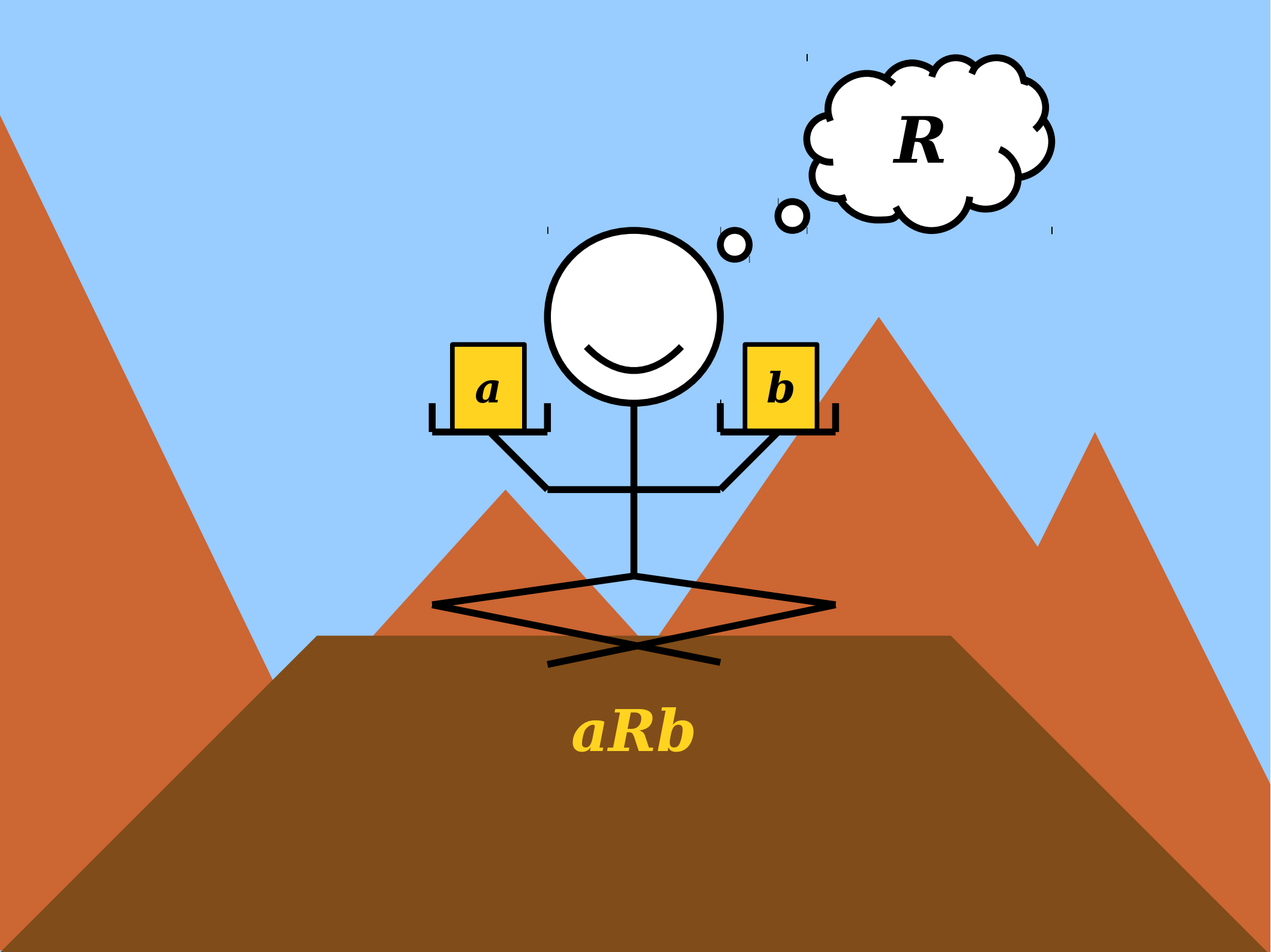
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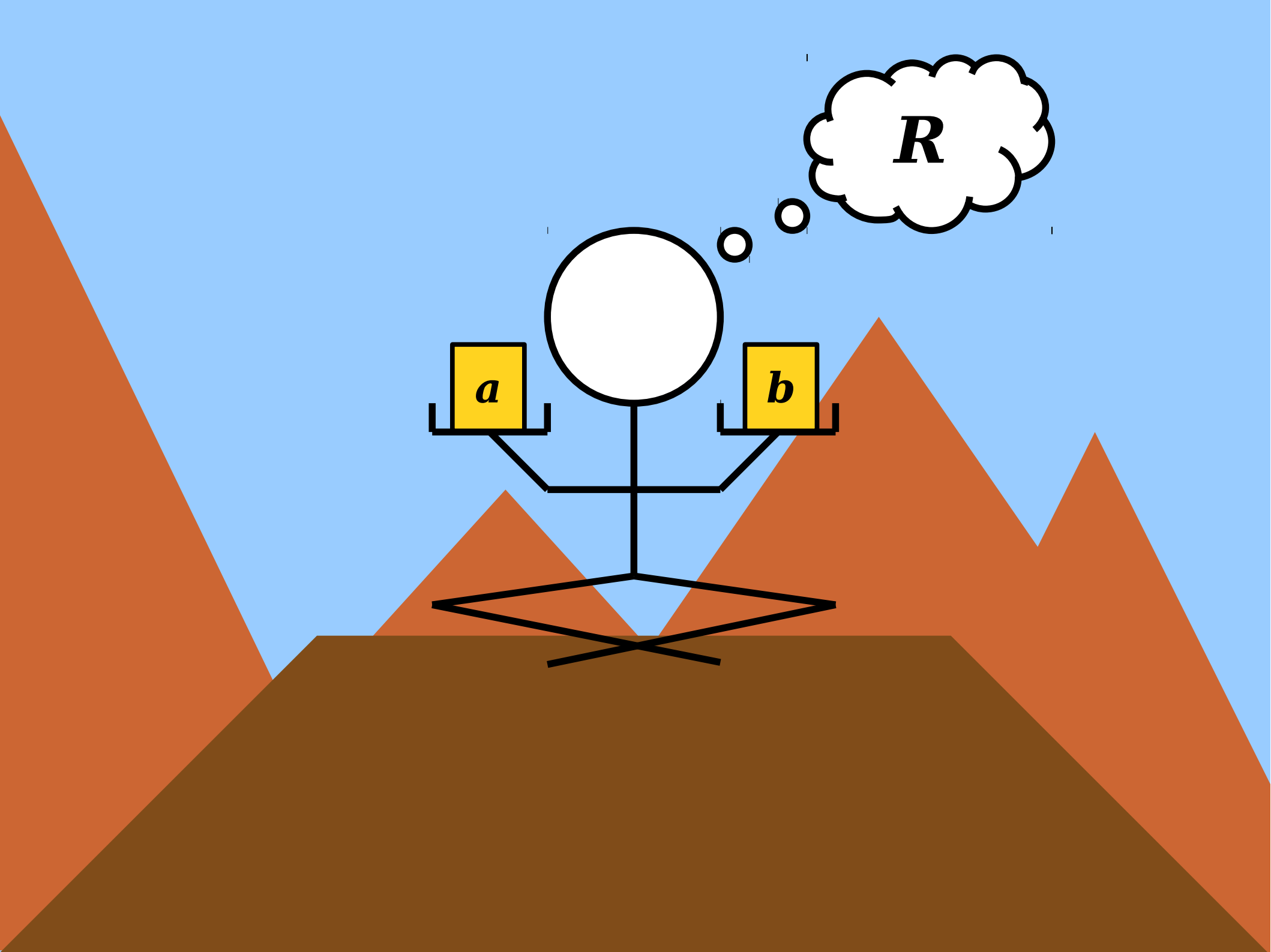
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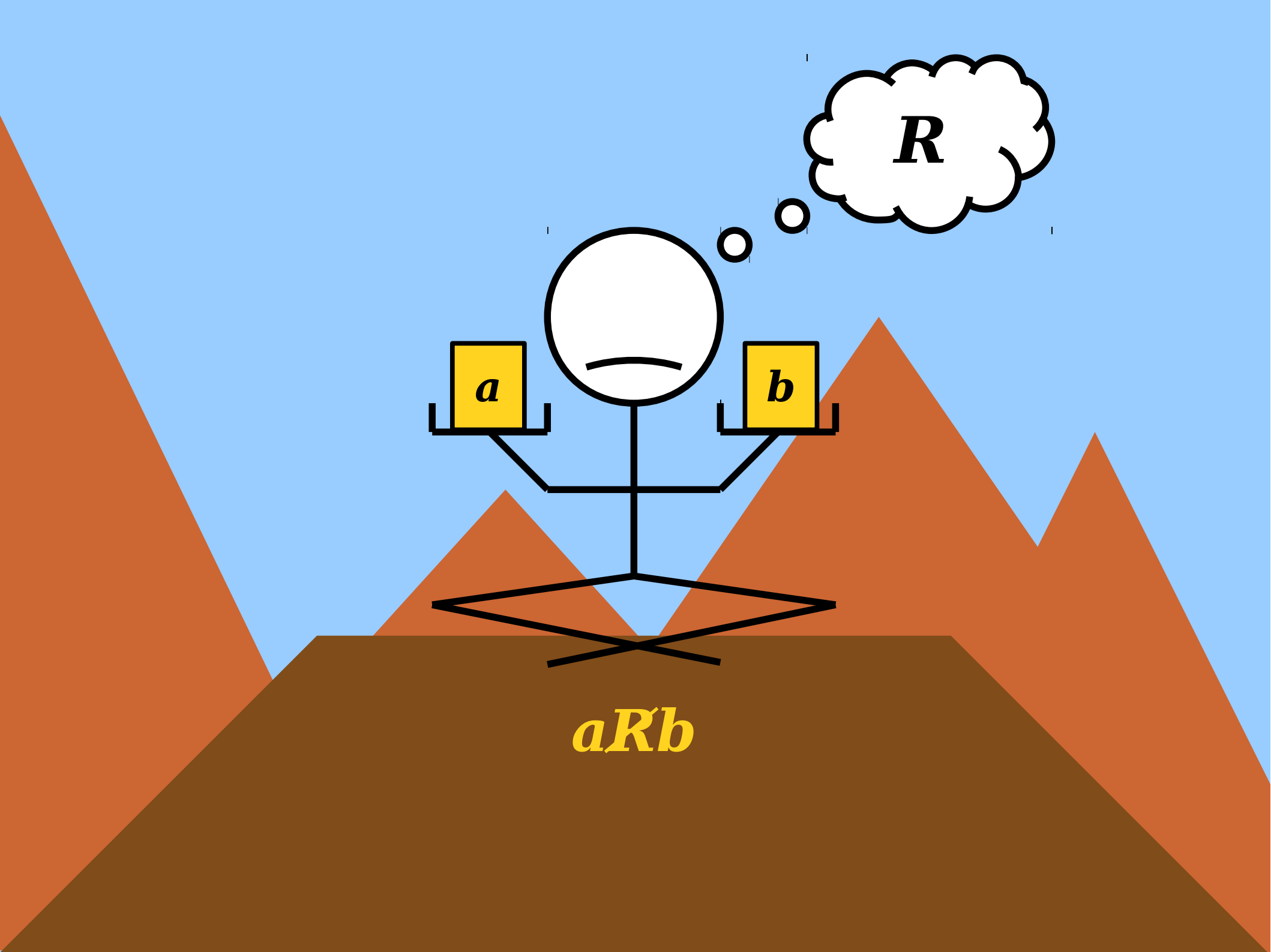
R
a
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aRb

Binary Relations

- A ***binary relation over a set A*** is a predicate R that can be applied to pairs of elements drawn from A.
- If R is a binary relation over A and it holds for the pair (a, b), we write ***aRb***.

$$3 = 3 \qquad 5 < 7 \qquad \varnothing \subseteq \mathbb{N}$$

- If R is a binary relation over A and it does not hold for the pair (a, b), we write ***aR̸b***.

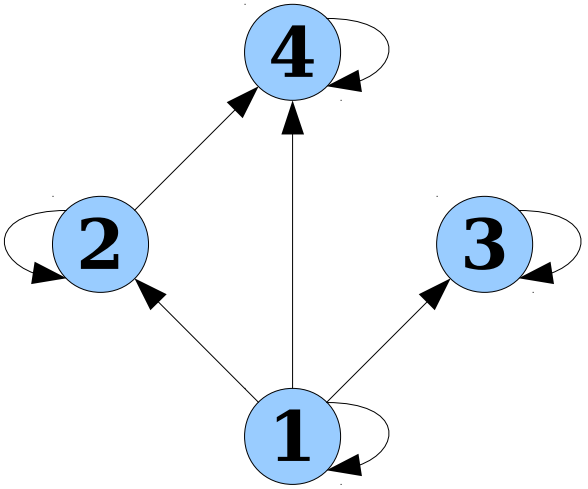
$$4 \neq 3 \qquad 4 \not< 3 \qquad \mathbb{N} \not\subseteq \varnothing$$

Properties of Relations

- Generally speaking, if R is a binary relation over a set A, the order of the operands *is significant*.
  - For example, $3 < 5$, but $5 \not< 3$.
  - *There are some specific relations (e.g., equals) for which order is irrelevant—more on this later!*
- Relations are always defined relative to some underlying set.
  - It's not meaningful to ask whether ☺ ⊆ 15, for example, since ⊆ is defined over sets, not arbitrary objects.

Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: the relation $a \mid b$ (meaning "a divides b") over the set {1, 2, 3, 4} looks like this:



Visualizing Relations

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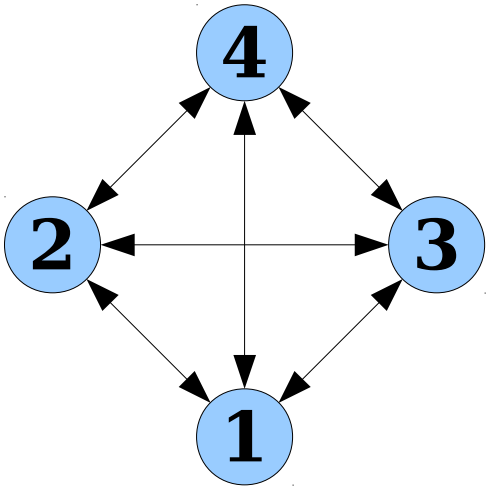
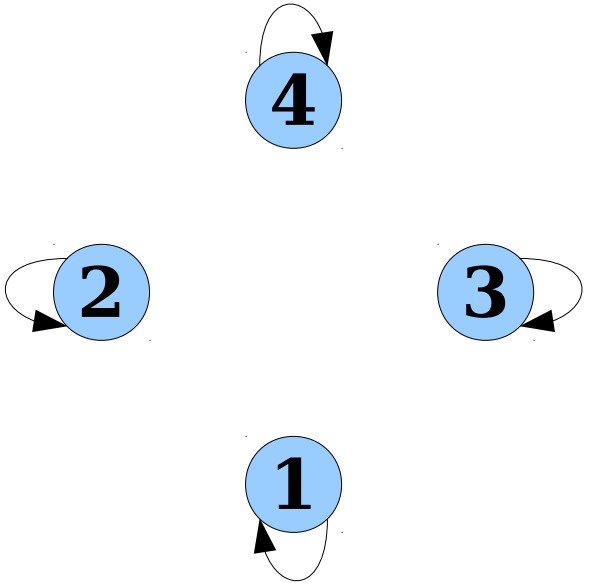
- Example: the relation $a \neq b$ over the set {1, 2, 3, 4} looks like this:



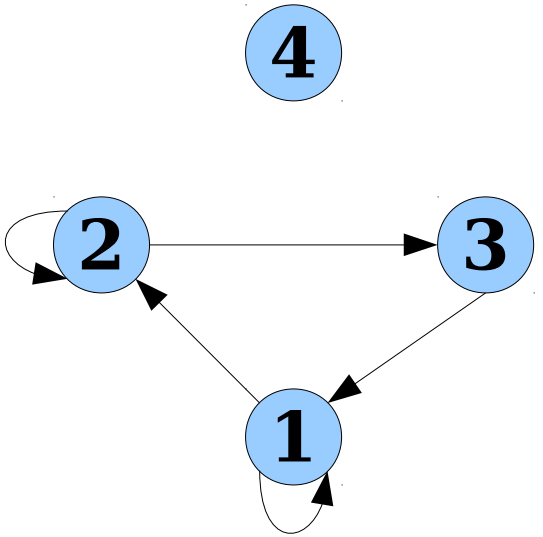
Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: the relation $a = b$ over the set {1, 2, 3, 4} looks like this:



Visualizing Relations

- We can visualize a binary relation R over a set A by drawing the elements of A and drawing a line between an element a and an element b if aRb is true.
- Example: below is some relation over {1, 2, 3, 4} that's a totally valid relation even though there doesn't appear to be a simple unifying rule.

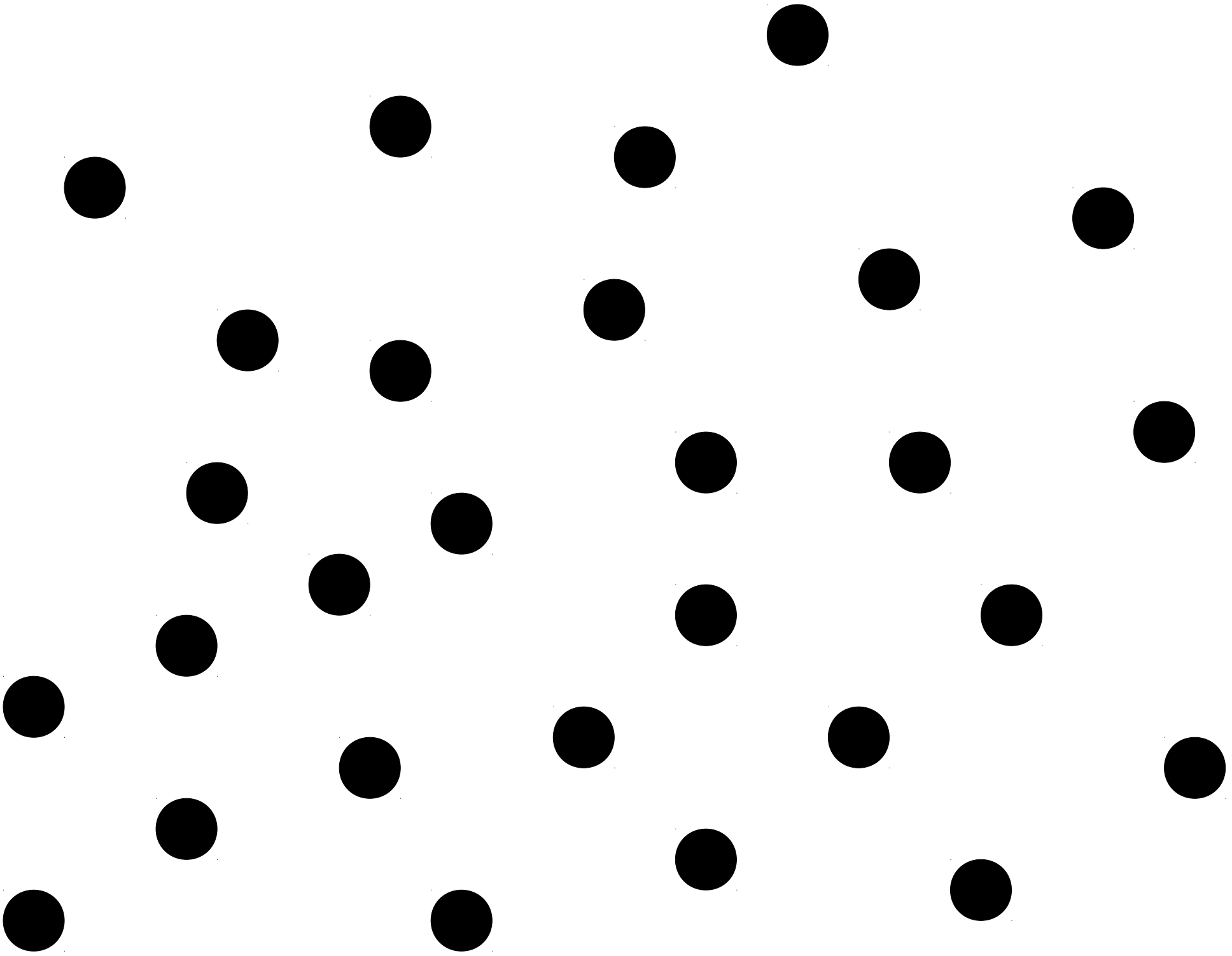


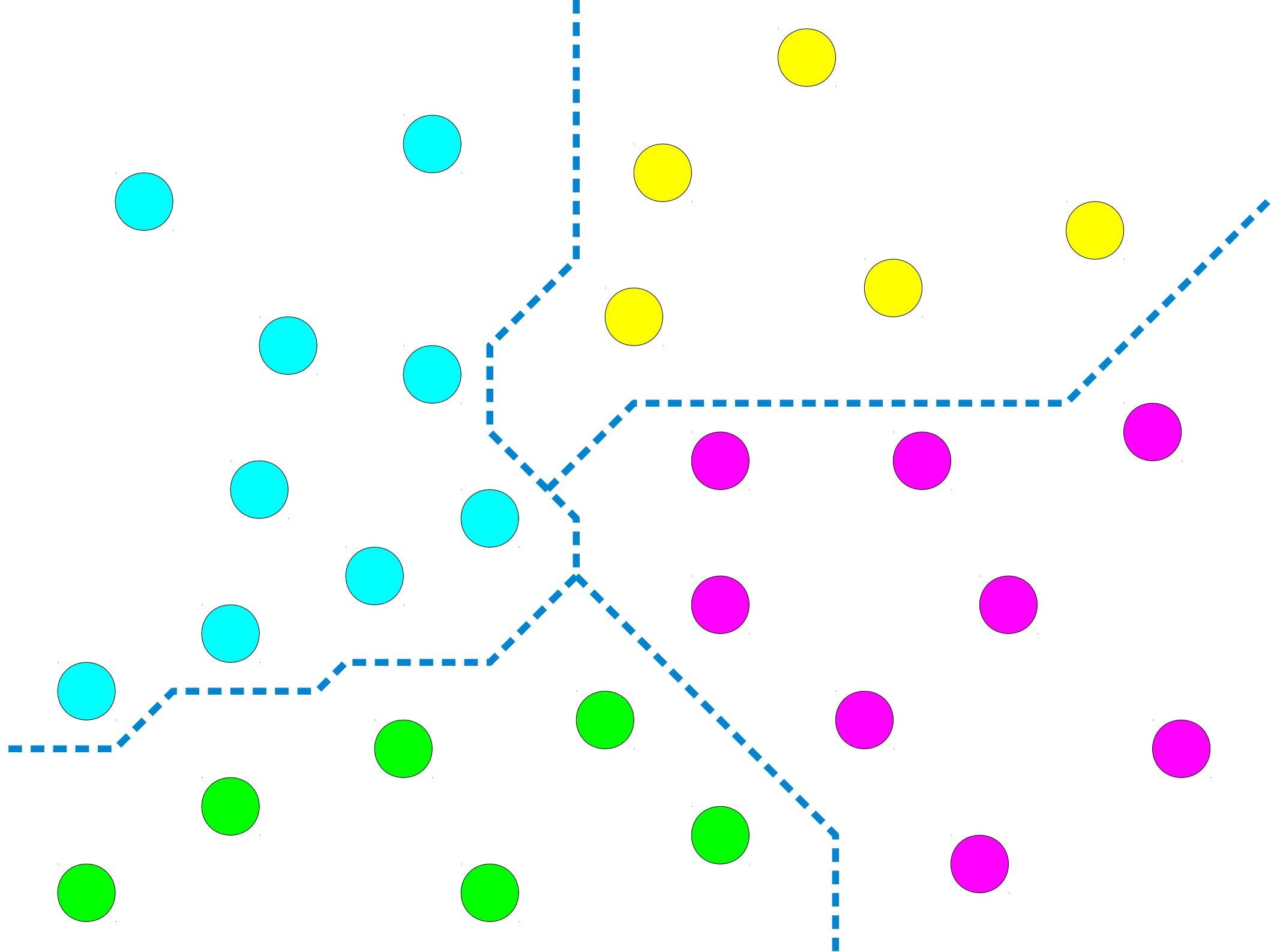
Capturing Structure

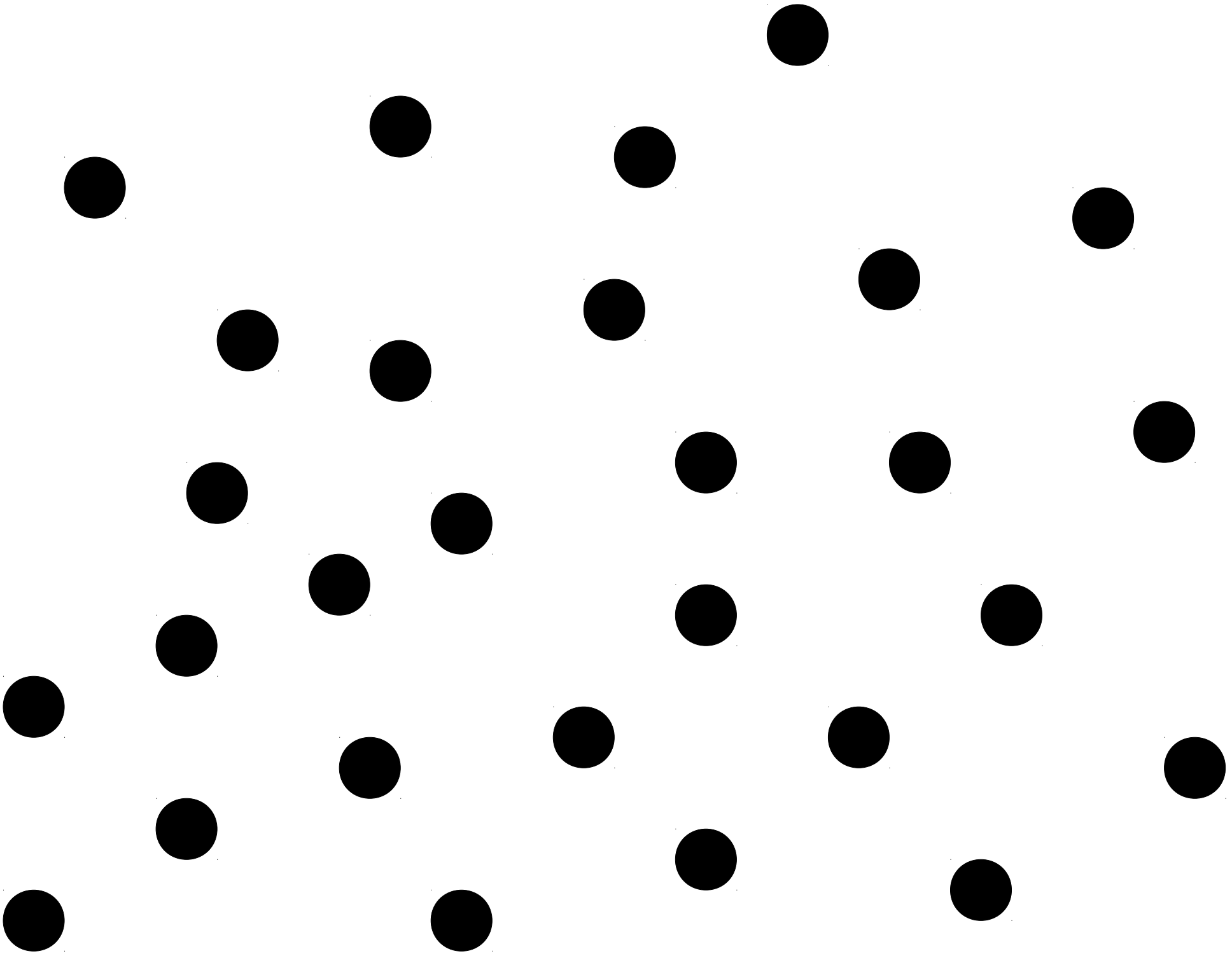
Capturing Structure

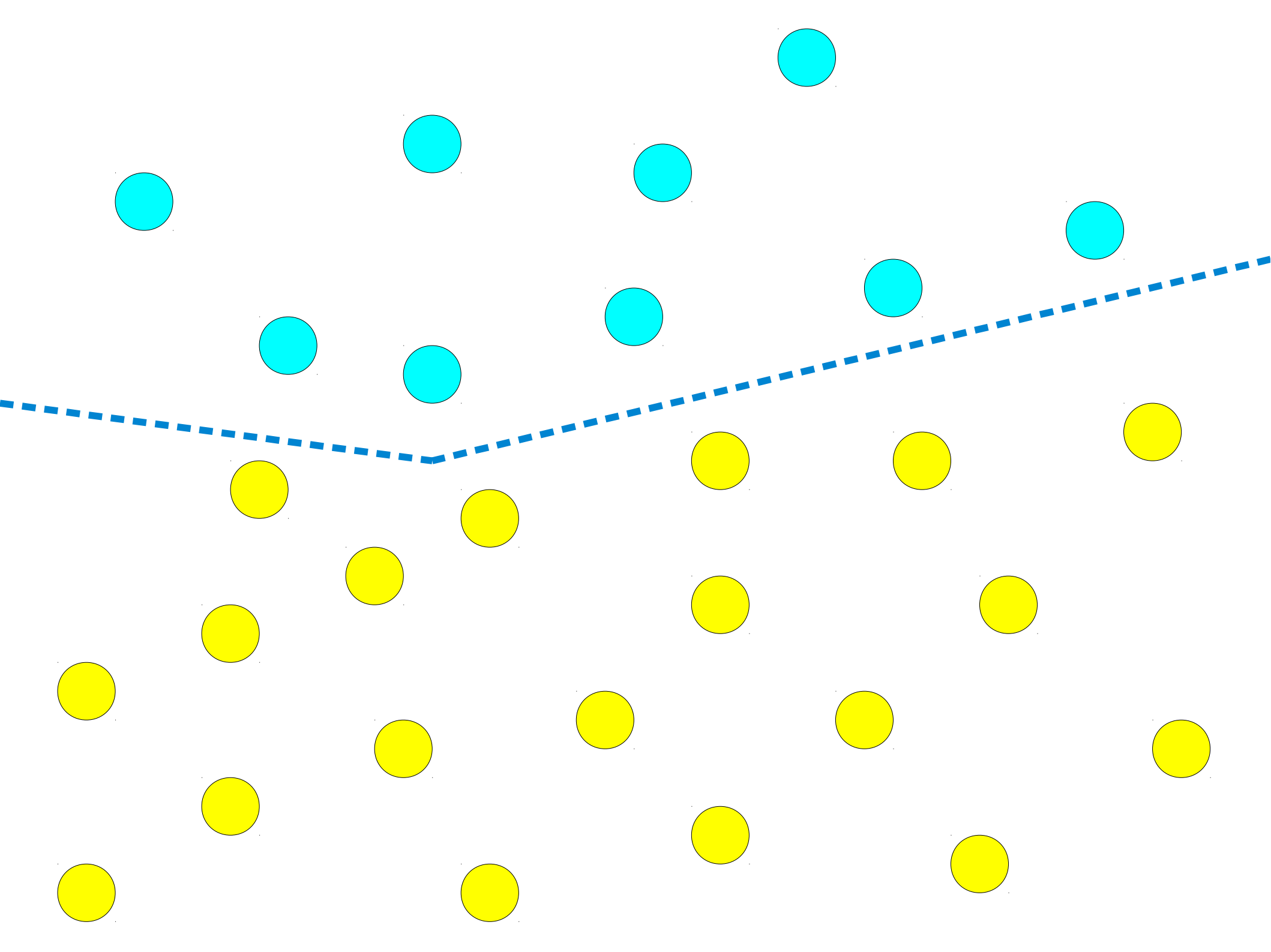
- Binary relations are an excellent way for capturing certain structures that appear in computer science.
- Today, we'll look at one of them (***partitions***).
- Along the way, we'll explore how to write proofs about definitions given in first-order logic.

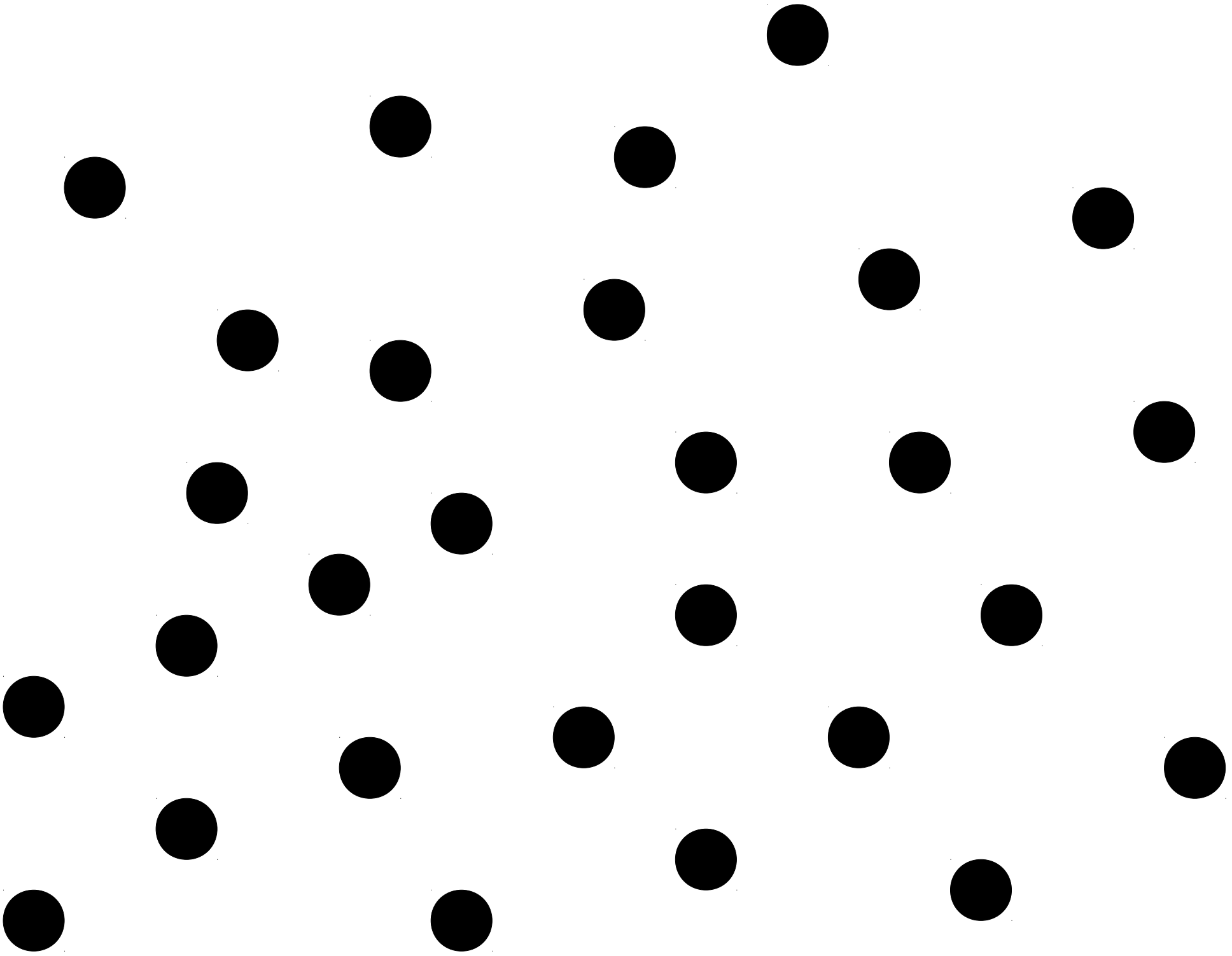
Partitions

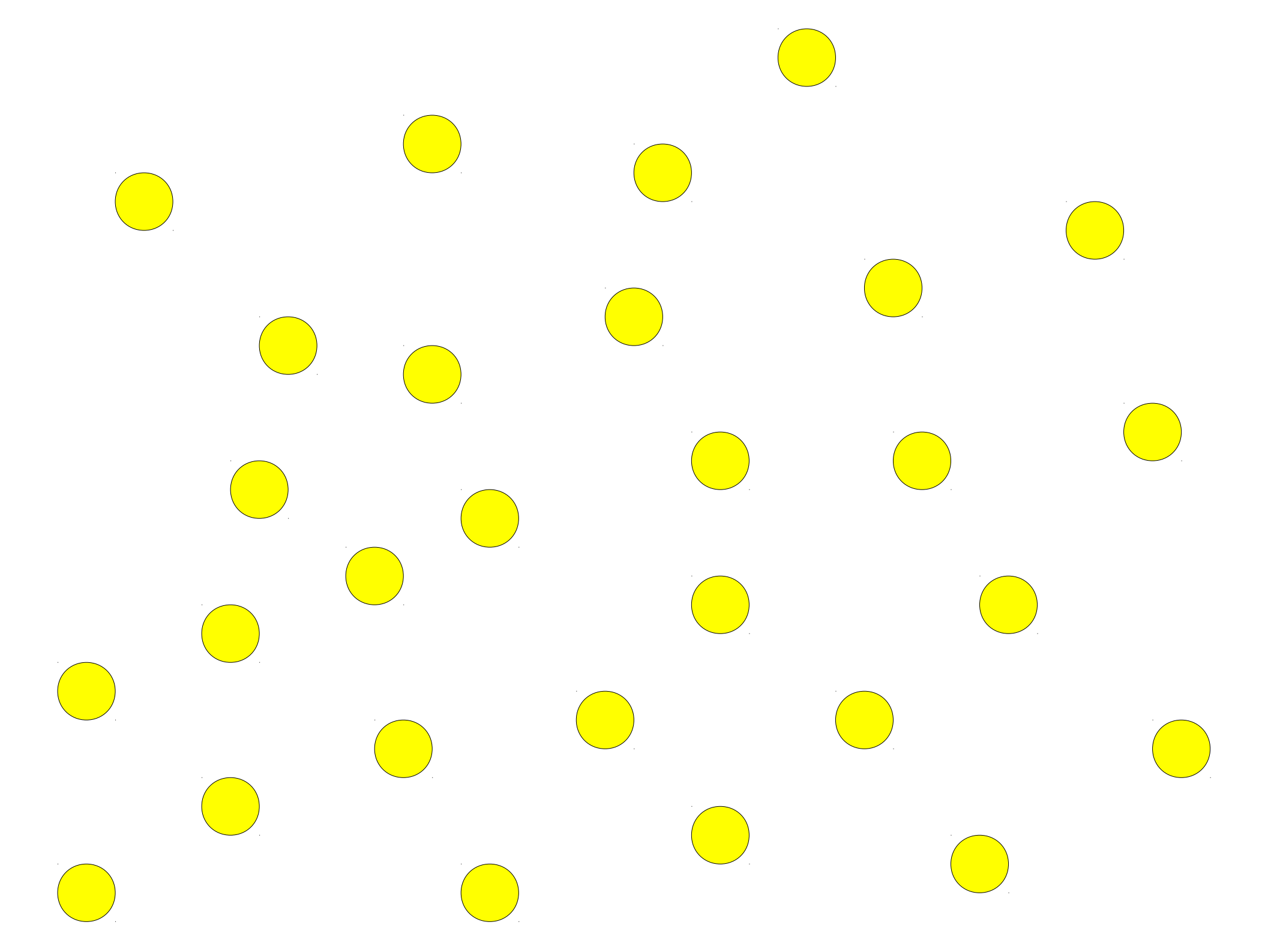


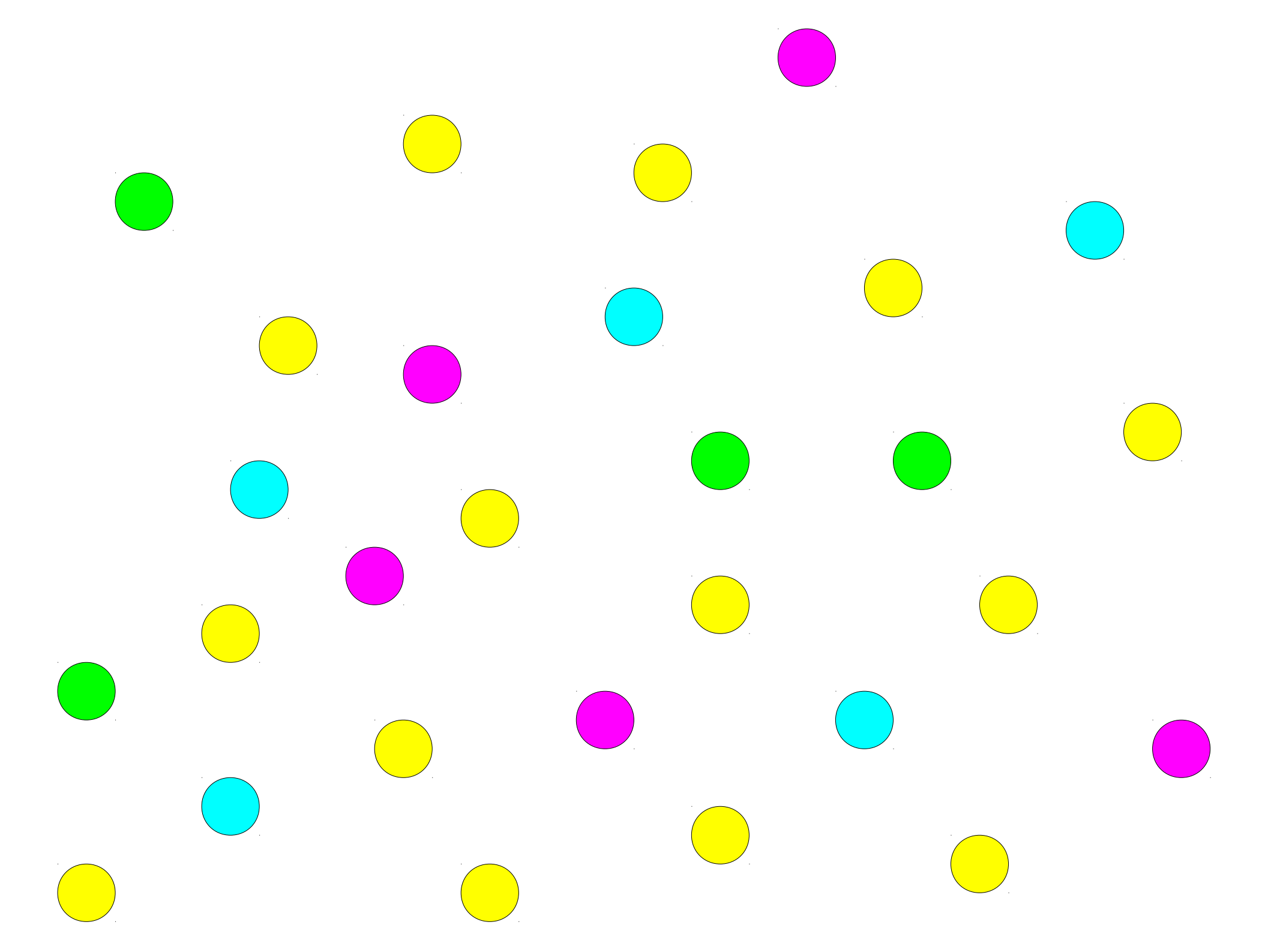








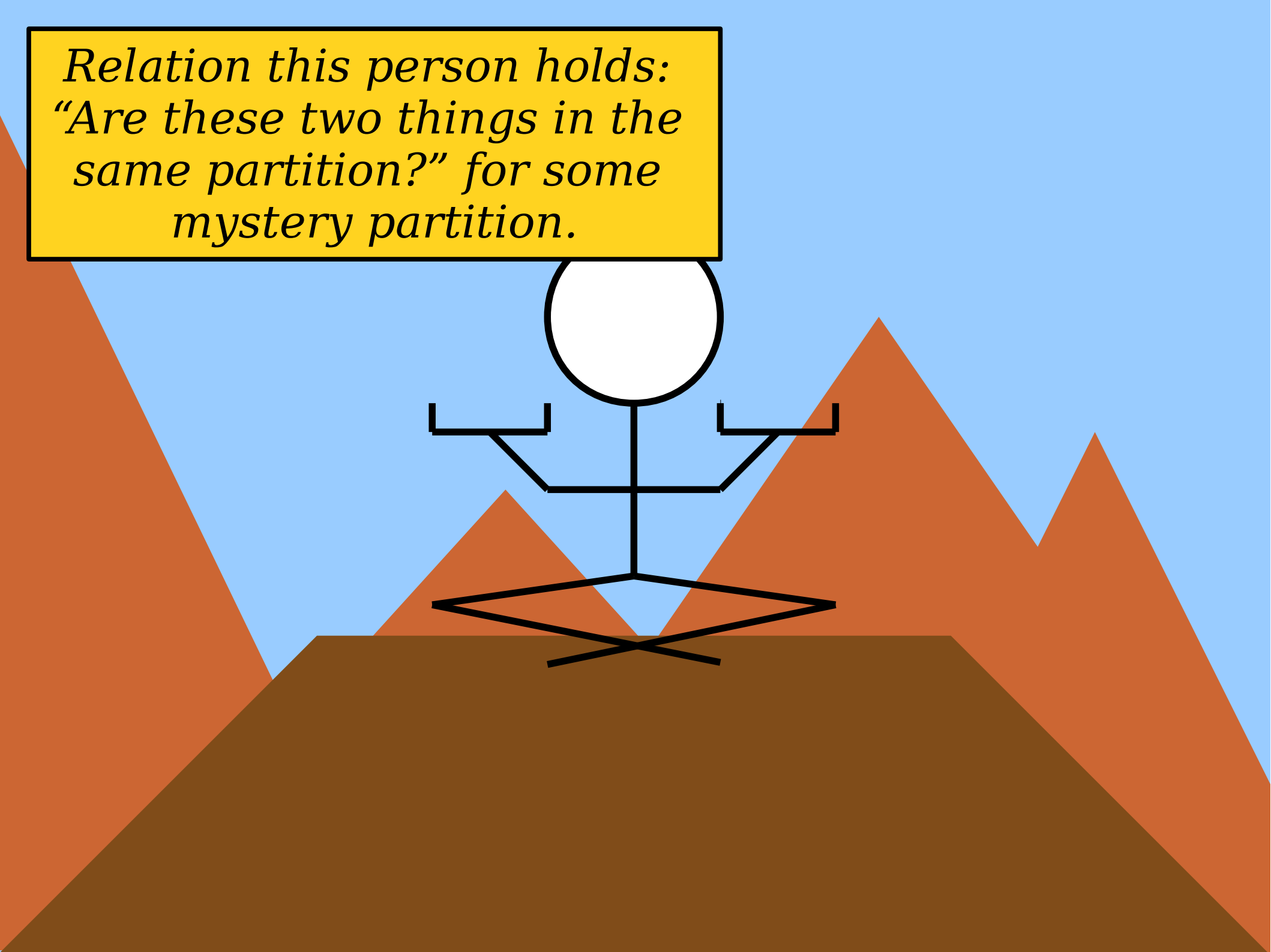




Partitions

- A ***partition of a set*** is a way of splitting the set into disjoint, nonempty subsets so that every element belongs to exactly one subset.
  - Two sets are ***disjoint*** if their intersection is the empty set; formally, sets S and T are disjoint if $S \cap T = \varnothing$.
- Intuitively, a partition of a set breaks the set apart into smaller pieces.
- There doesn't have to be any rhyme or reason to what those pieces are, though often there is one.

What's the connection between partitions and binary relations?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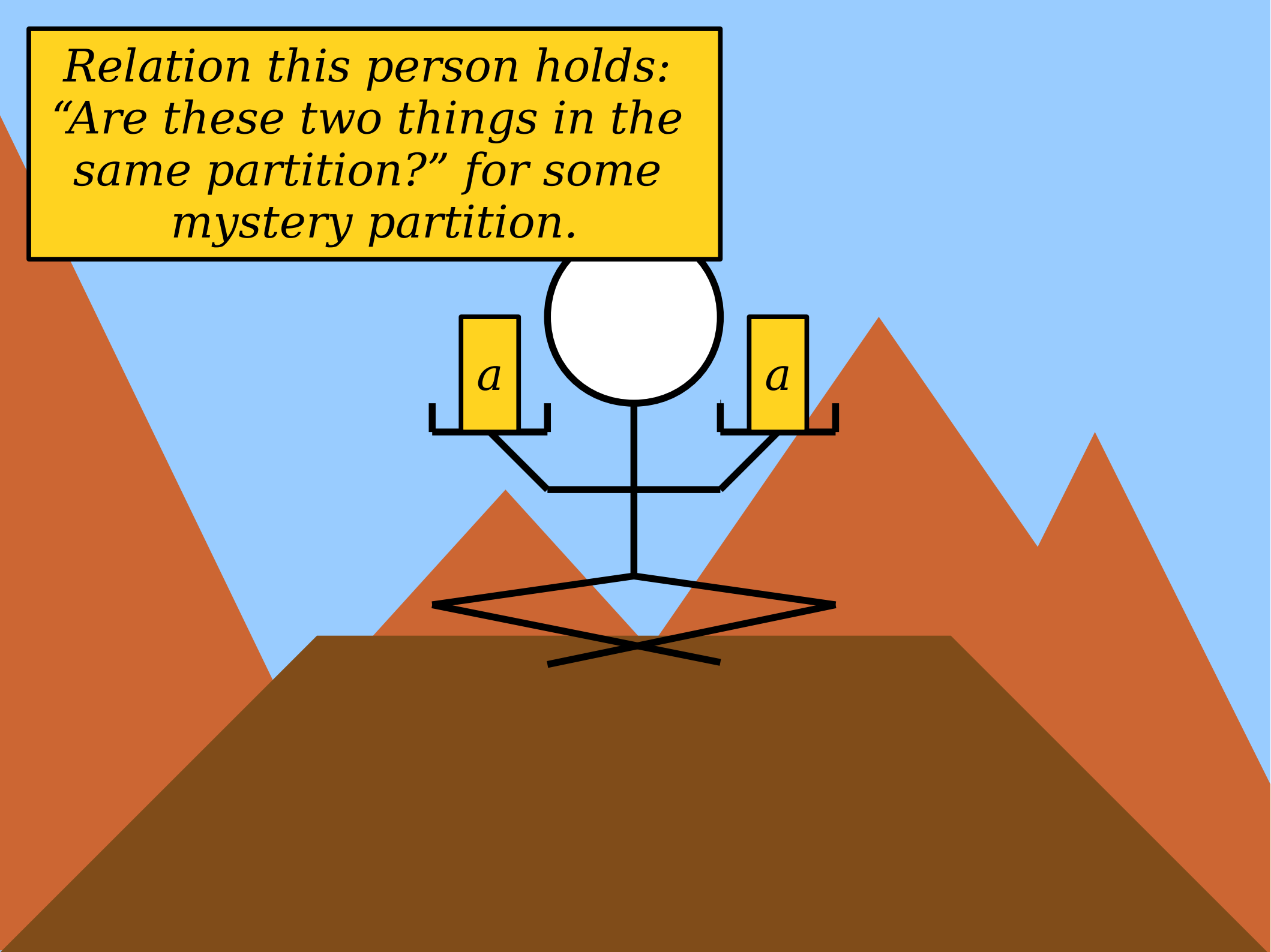

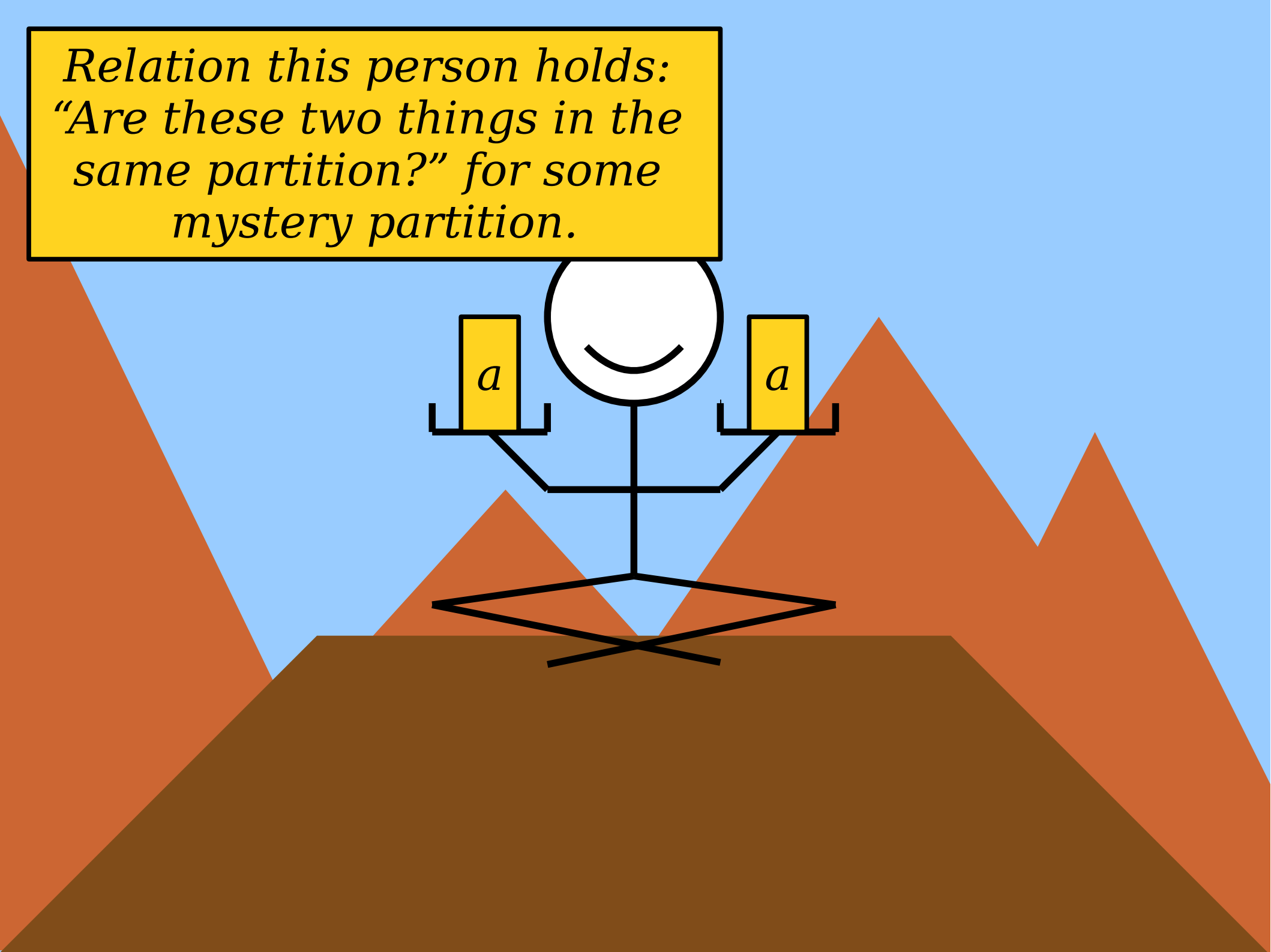
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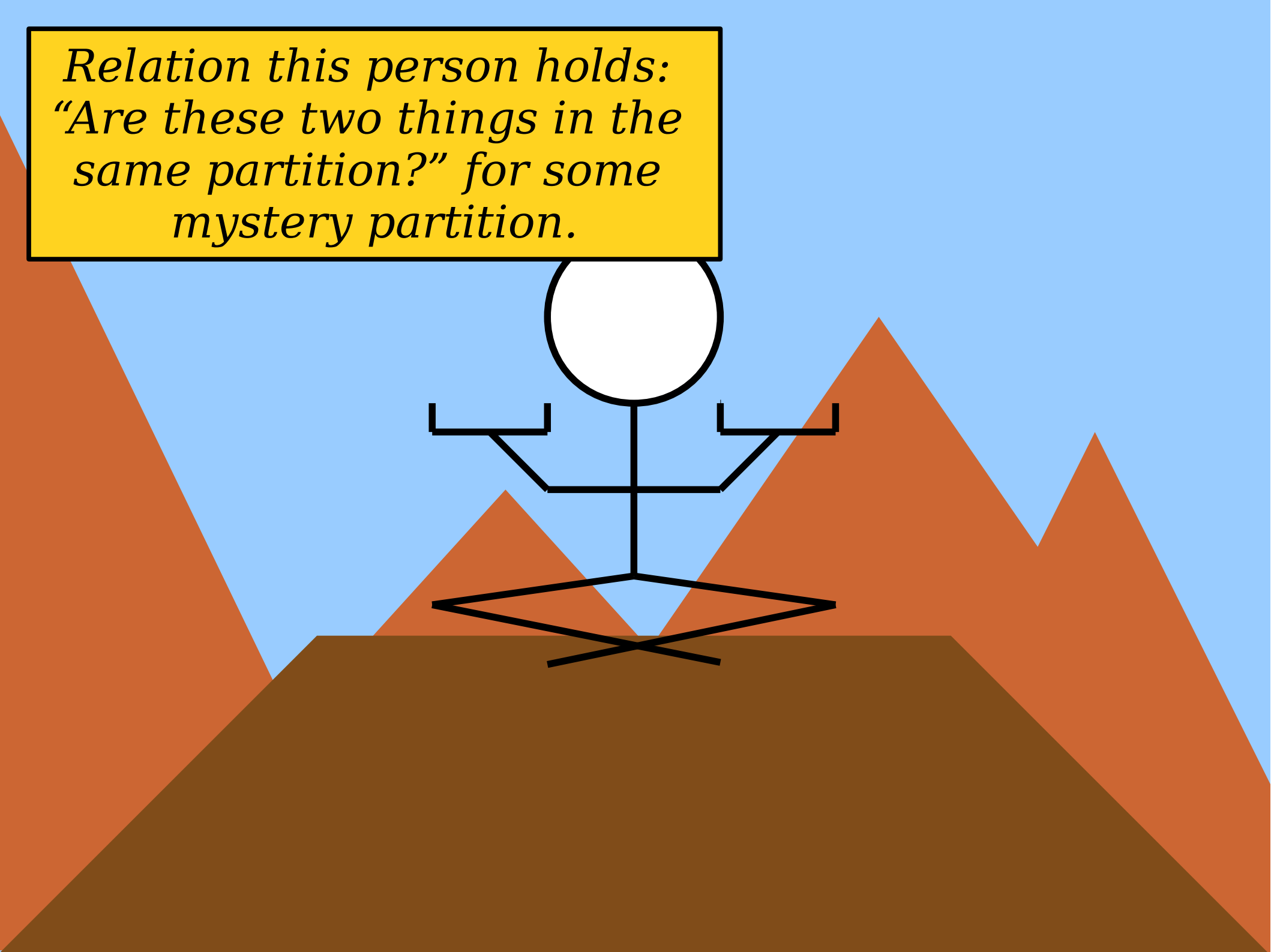
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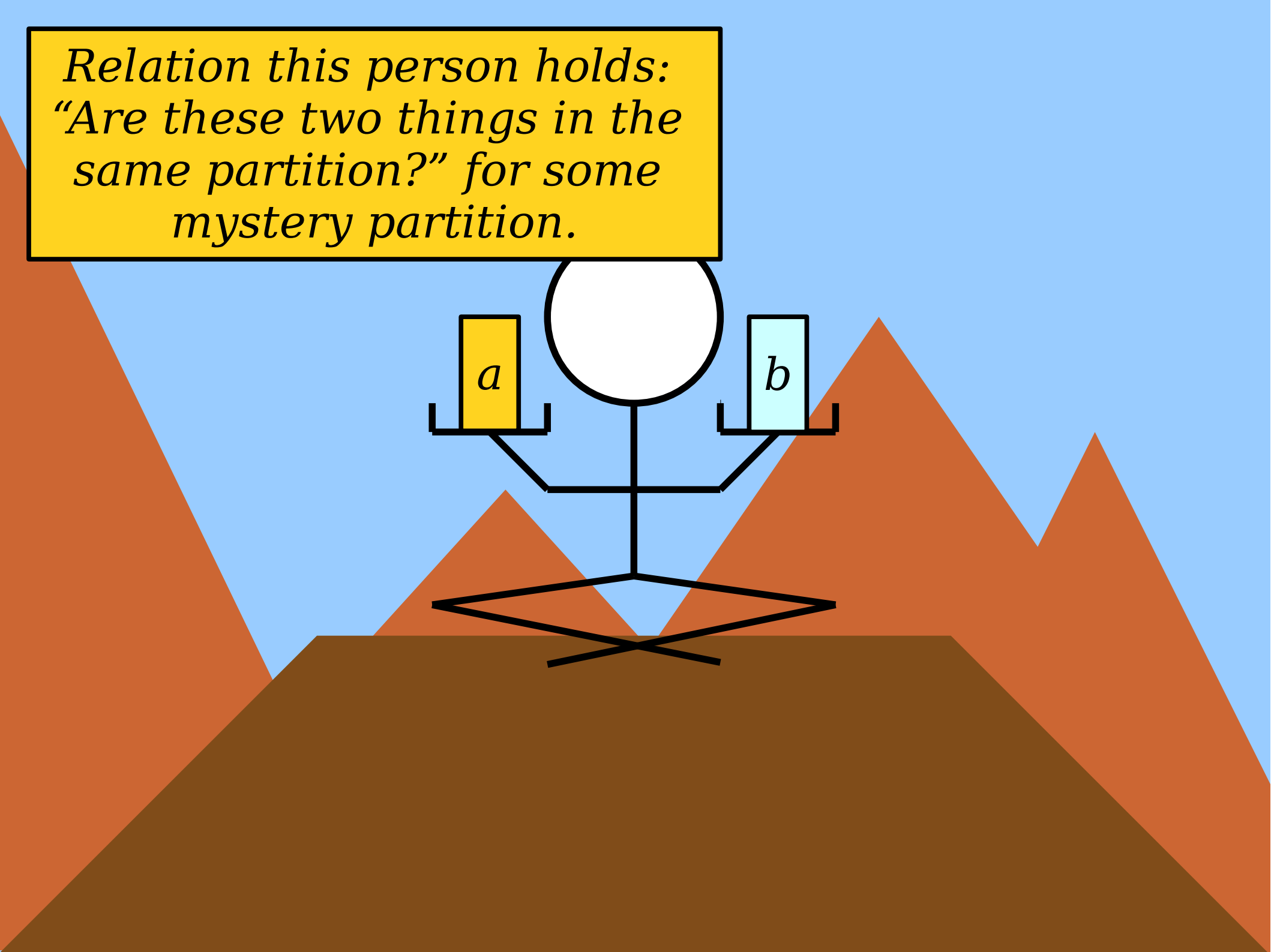

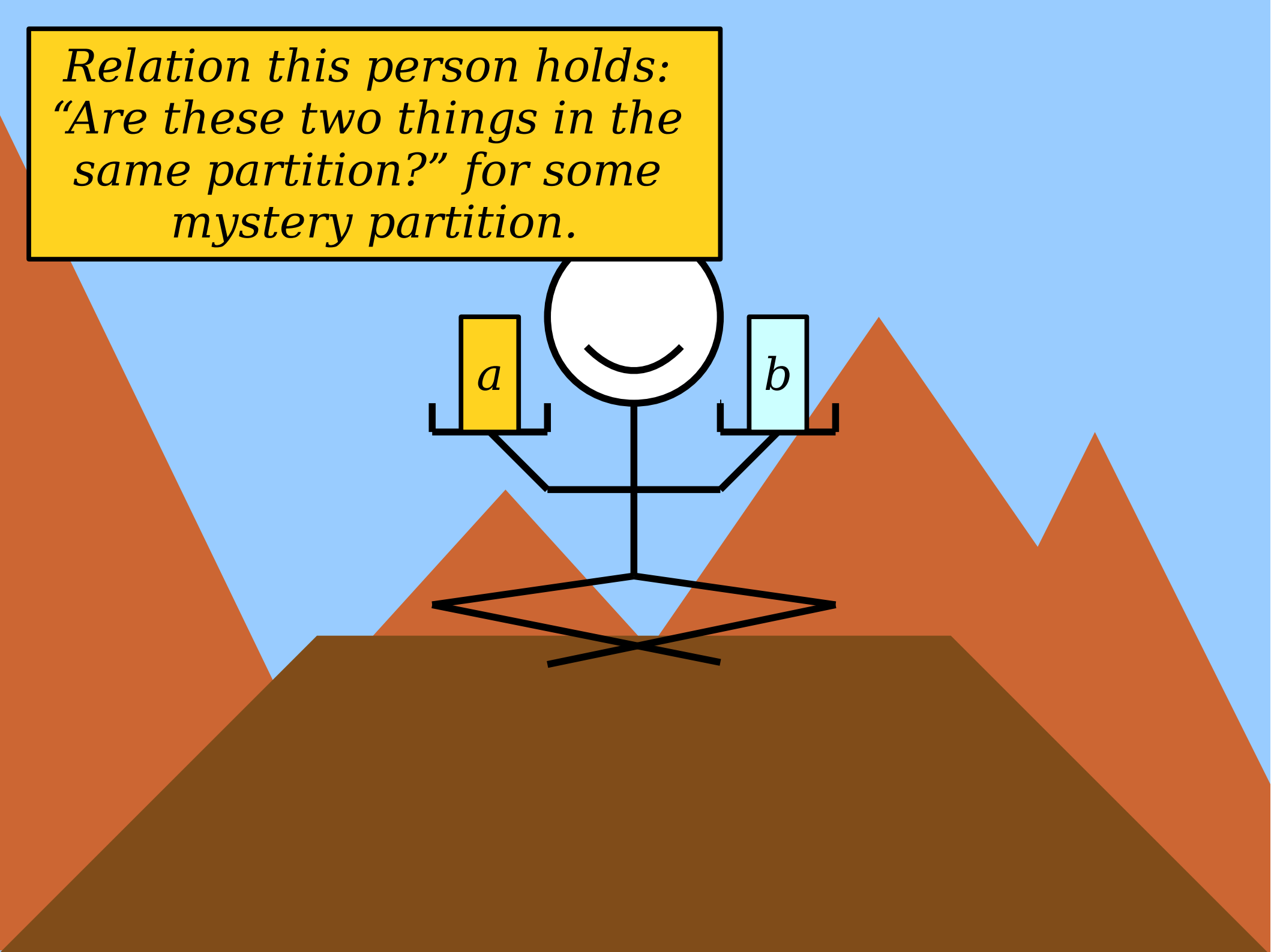
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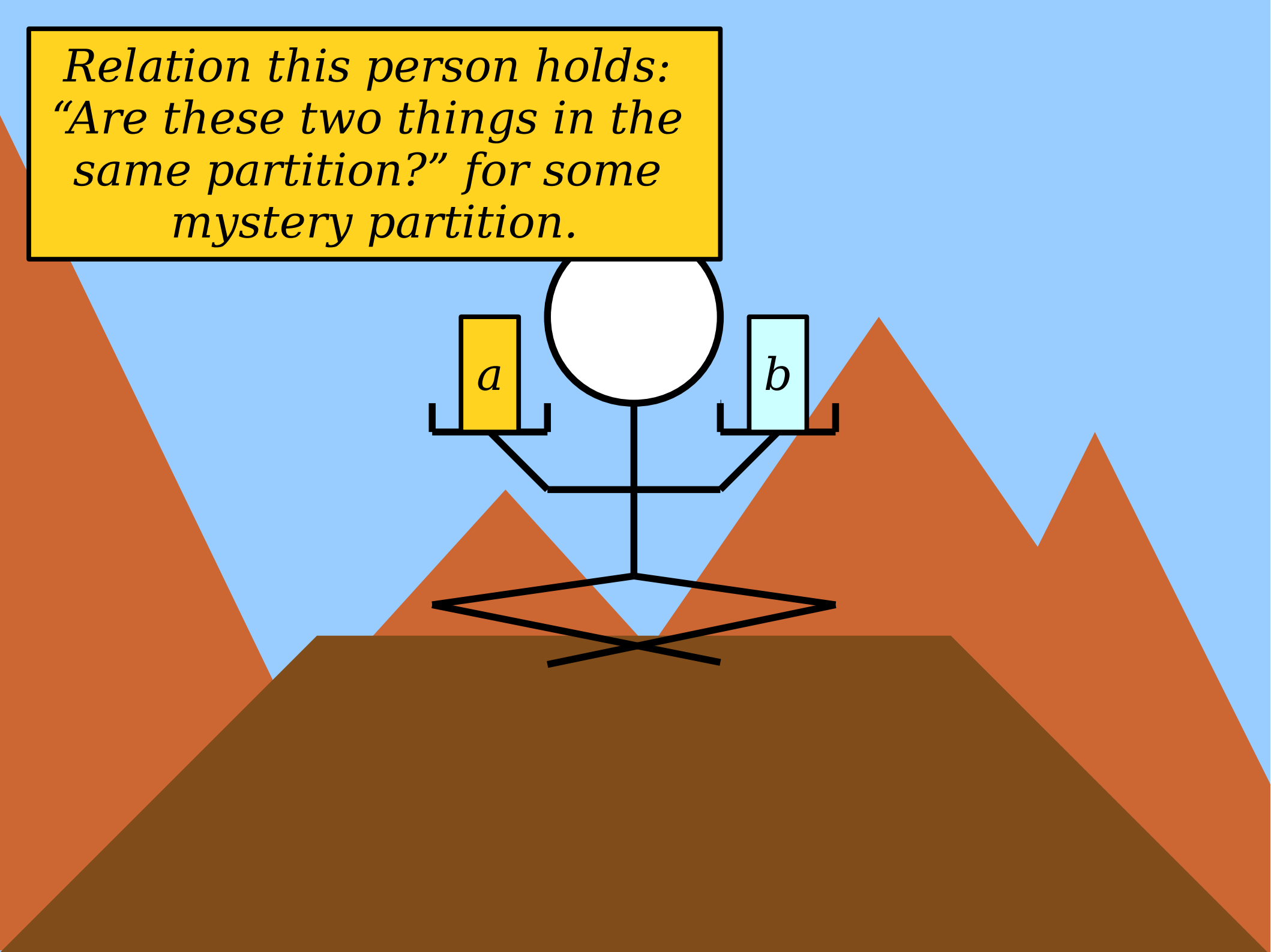

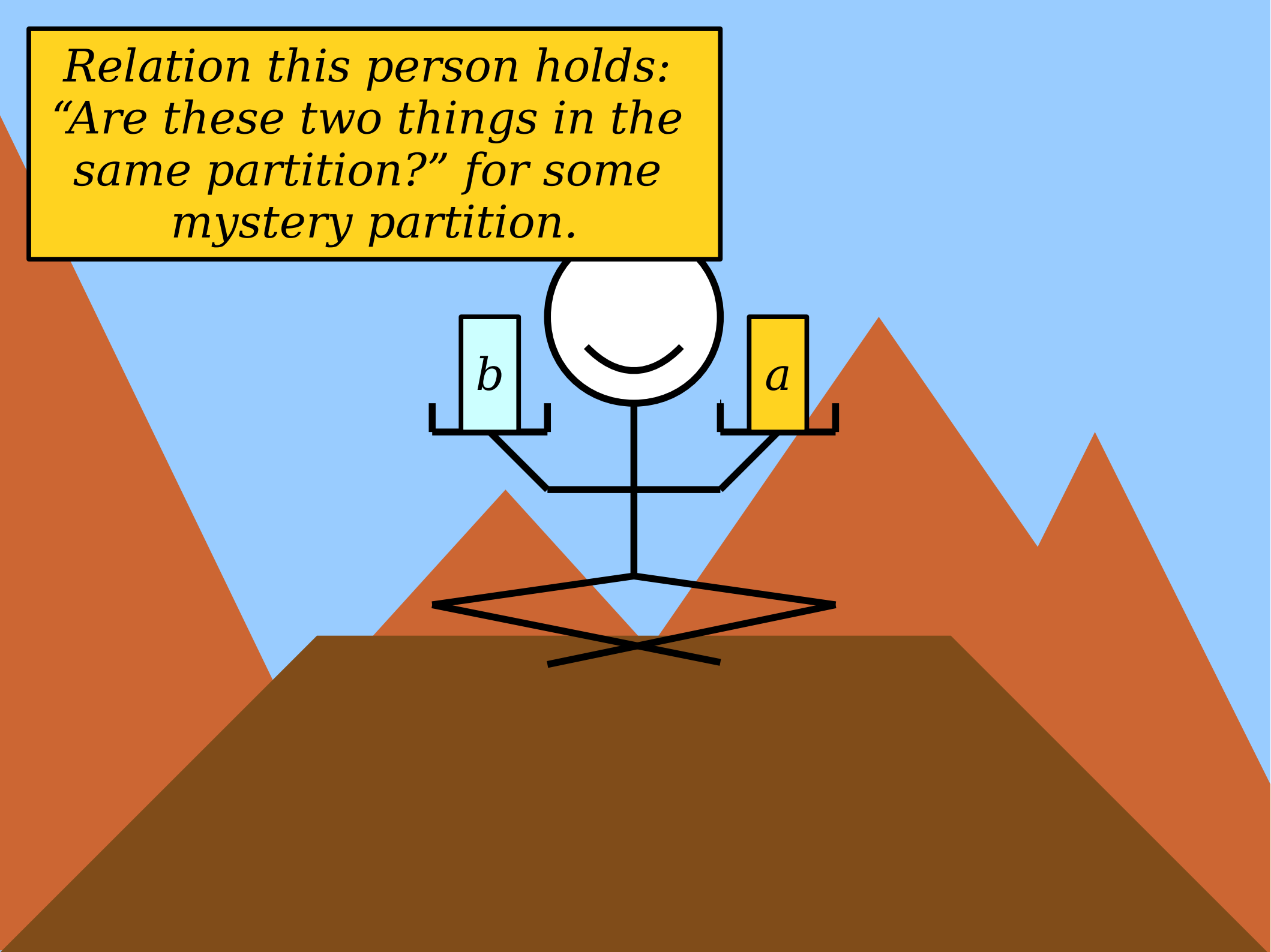
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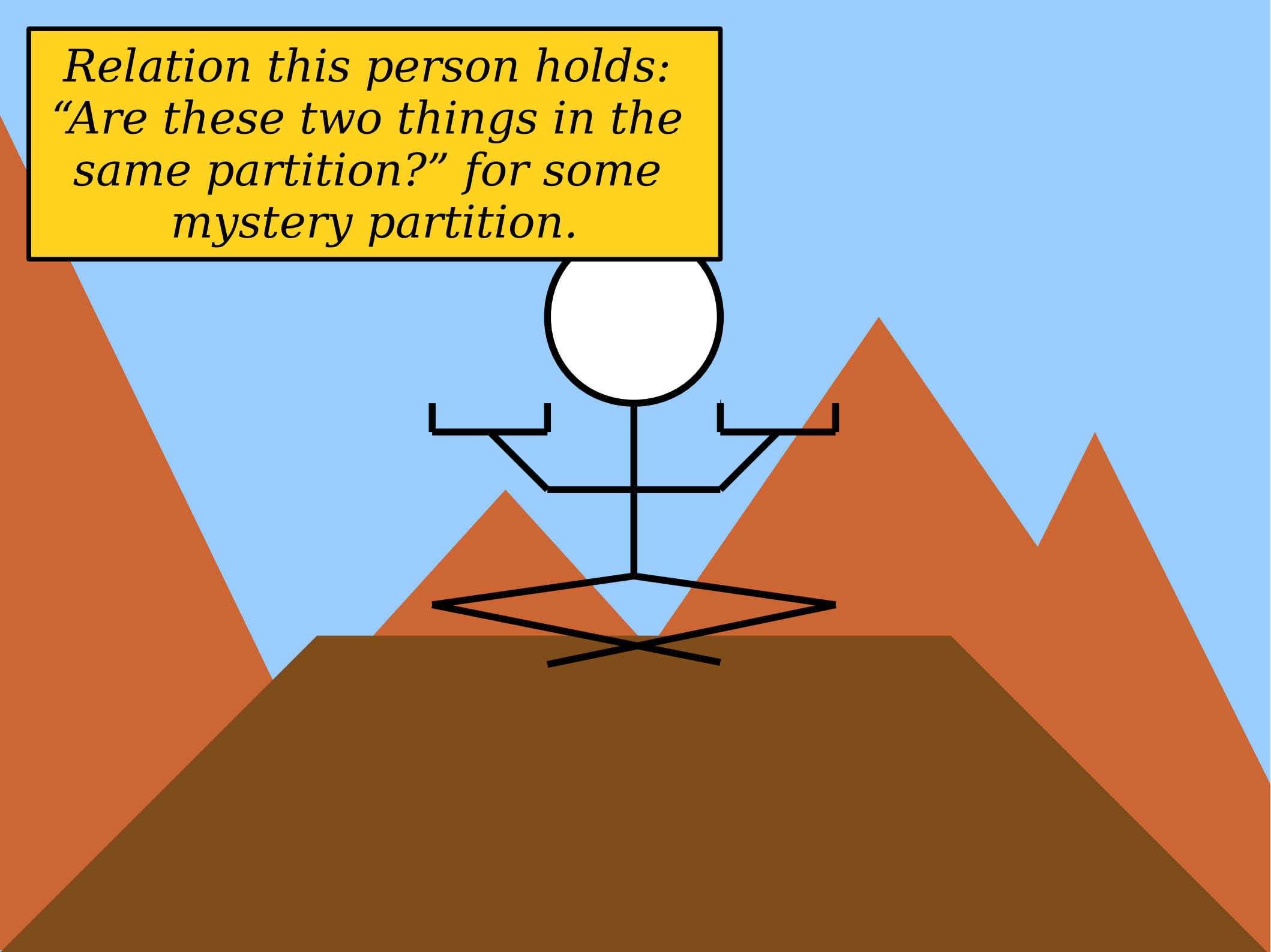
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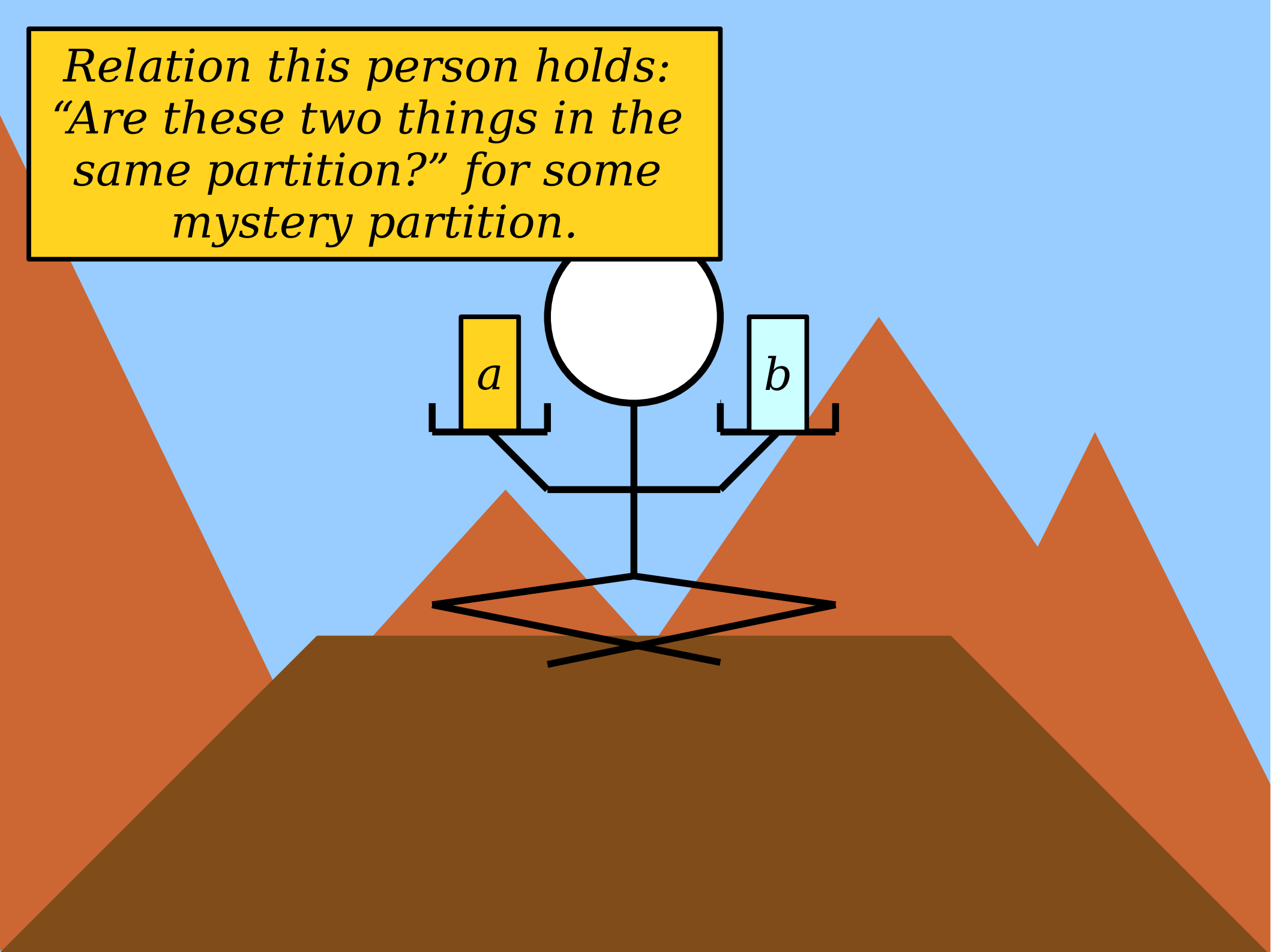

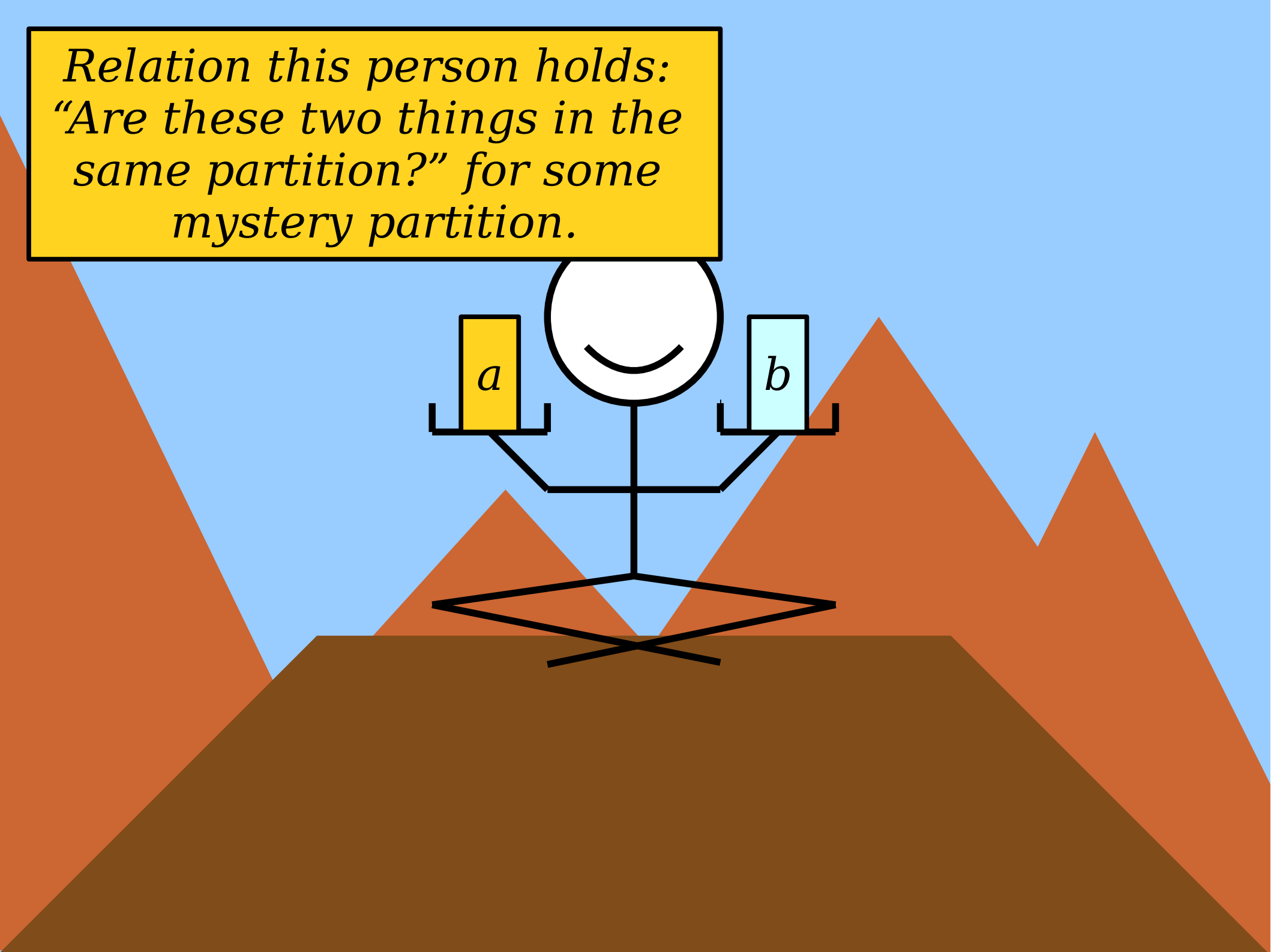
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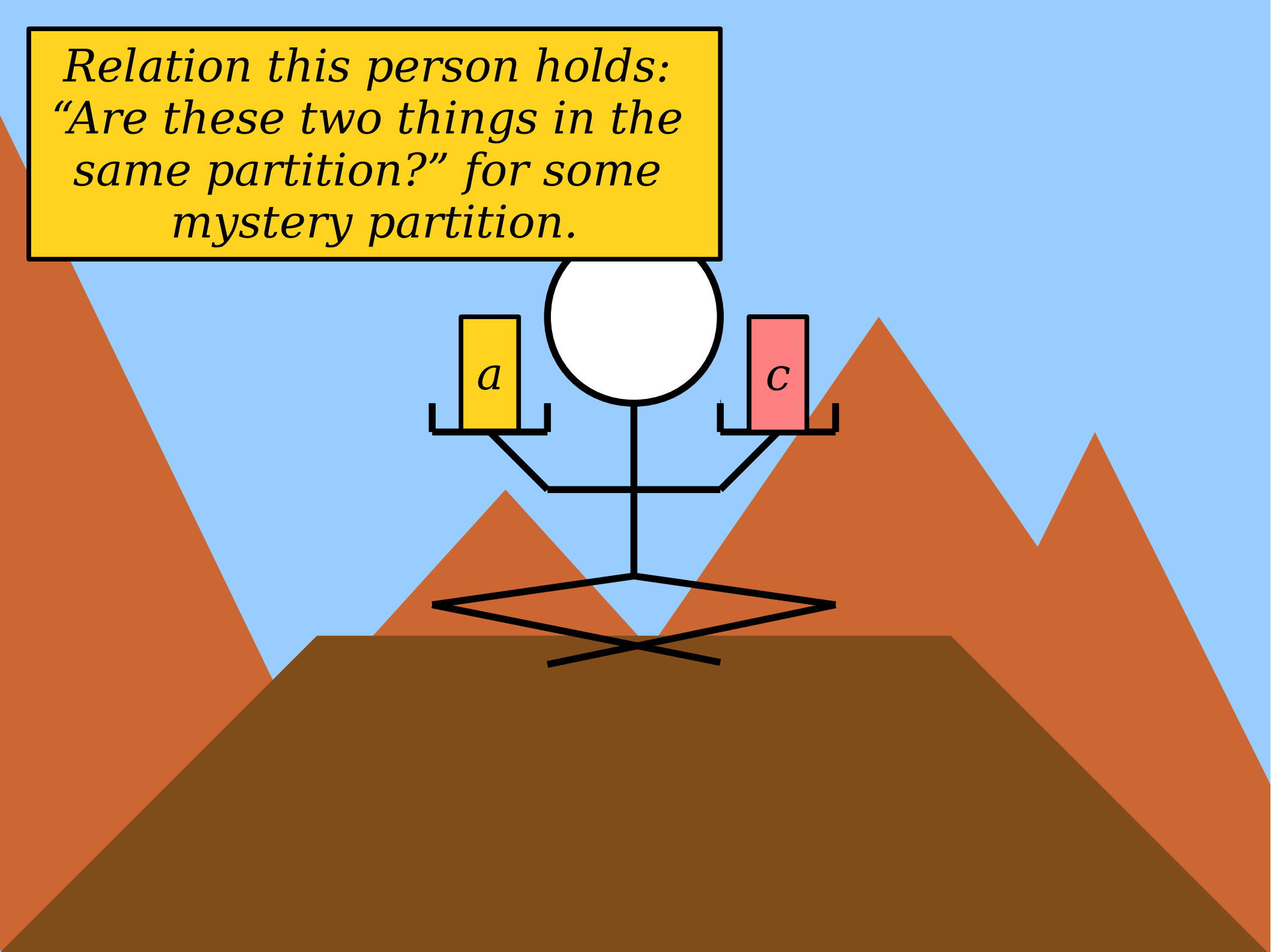

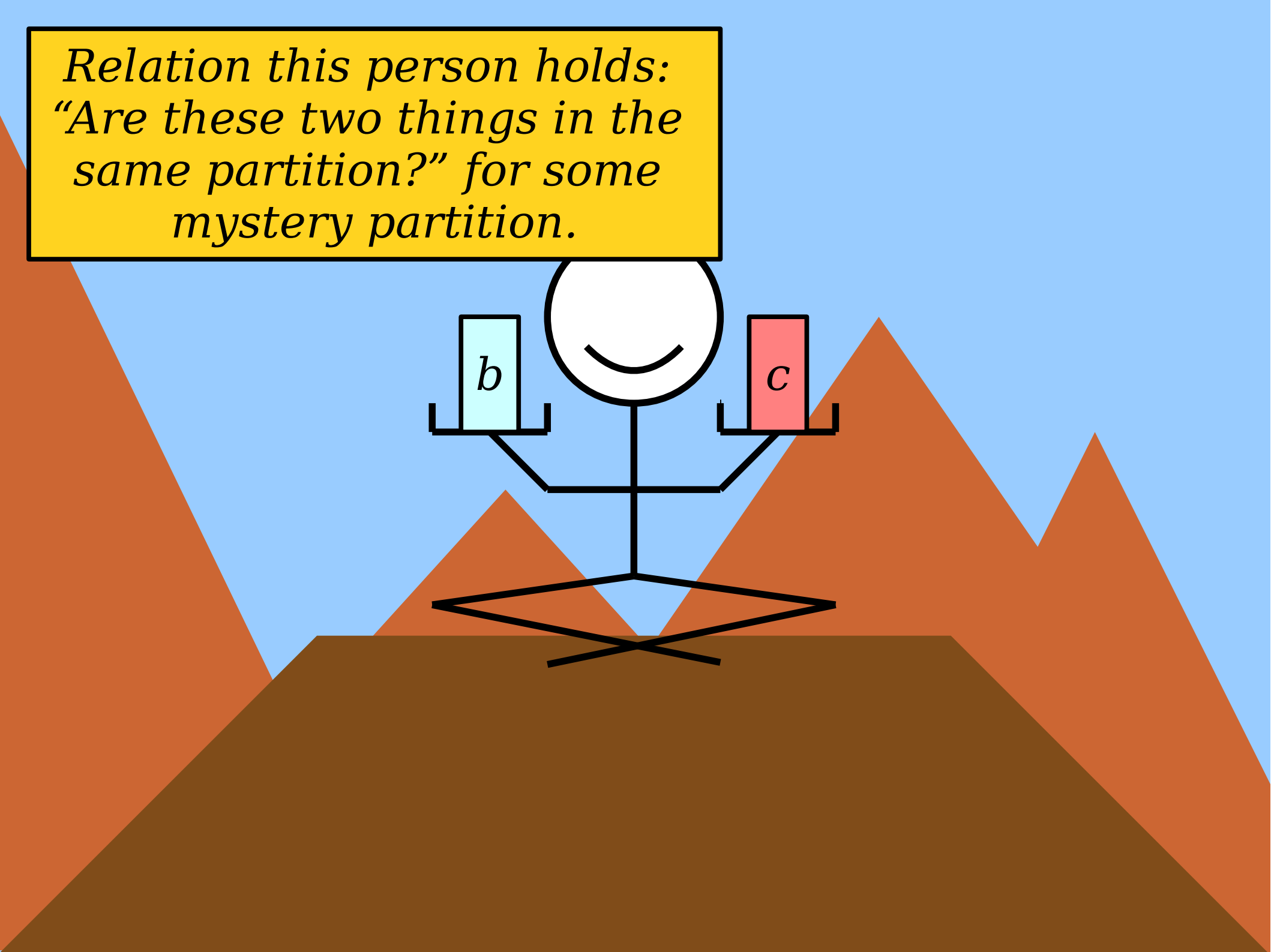
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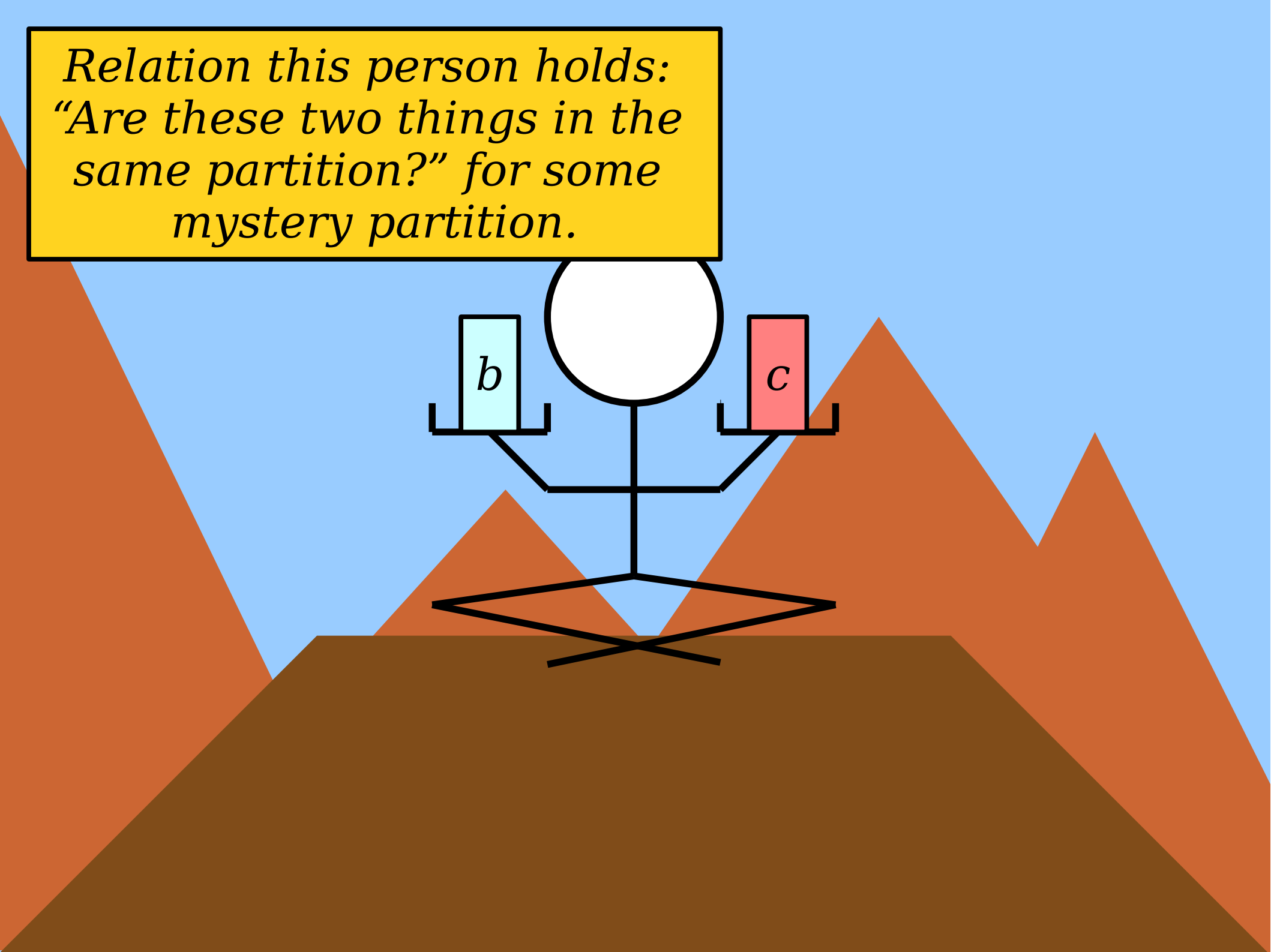

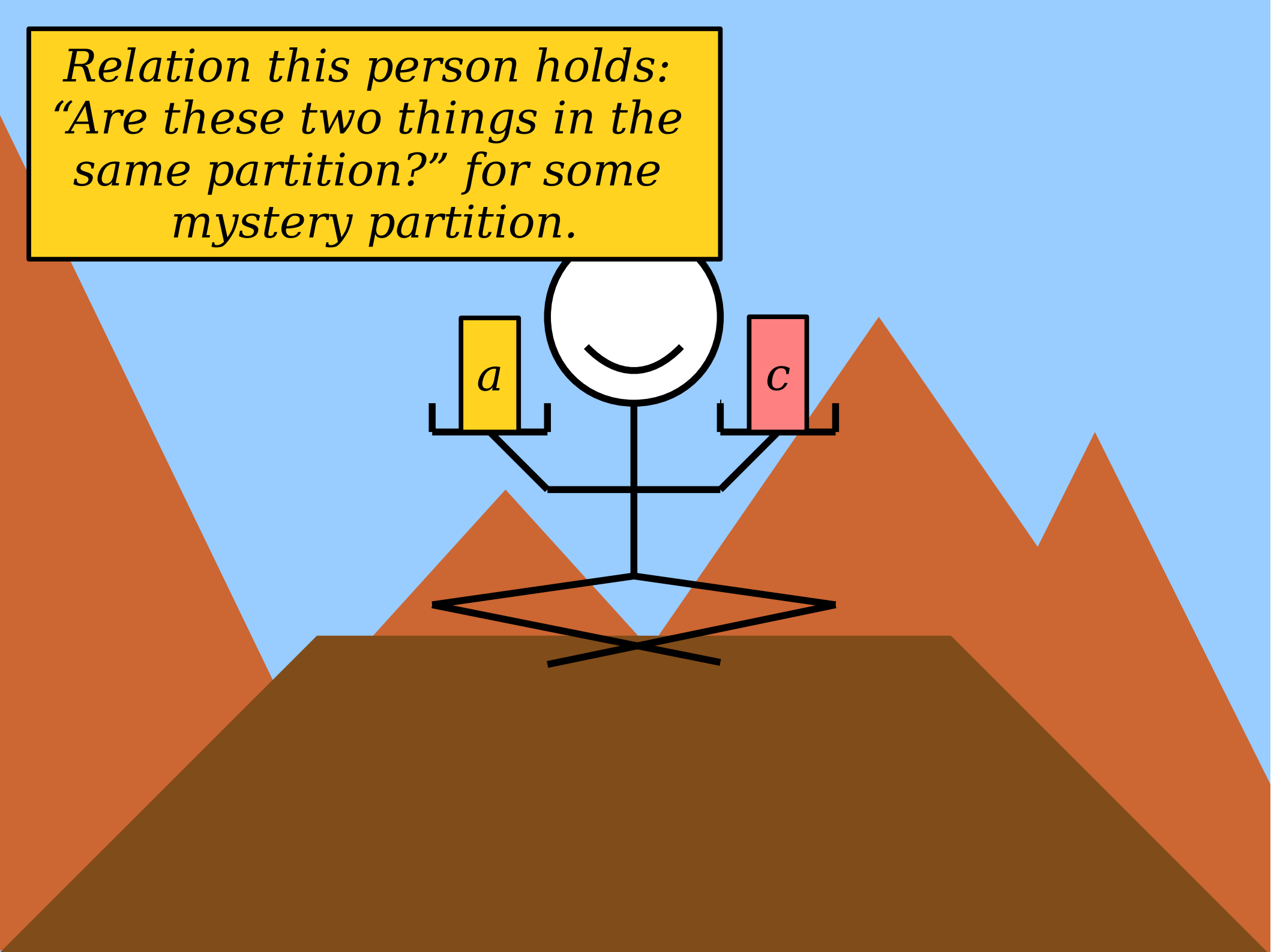
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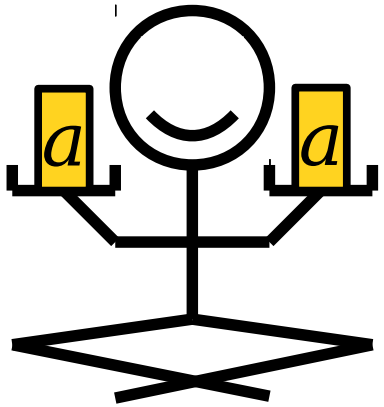

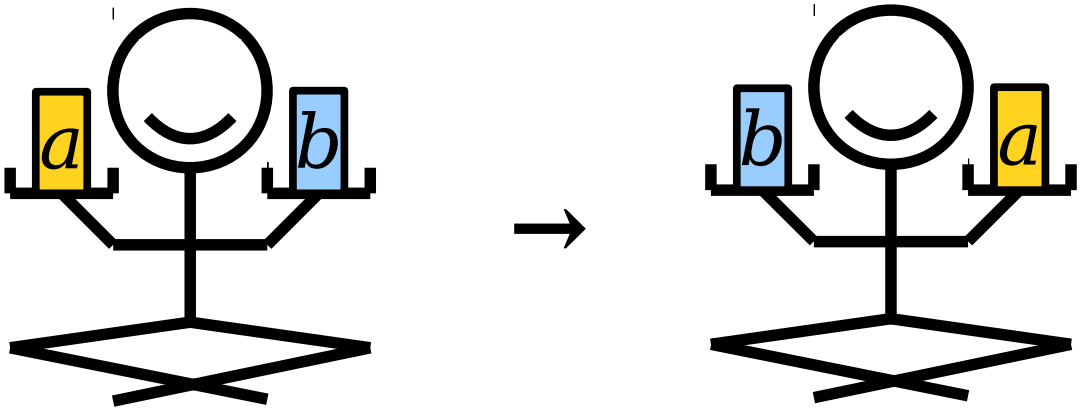

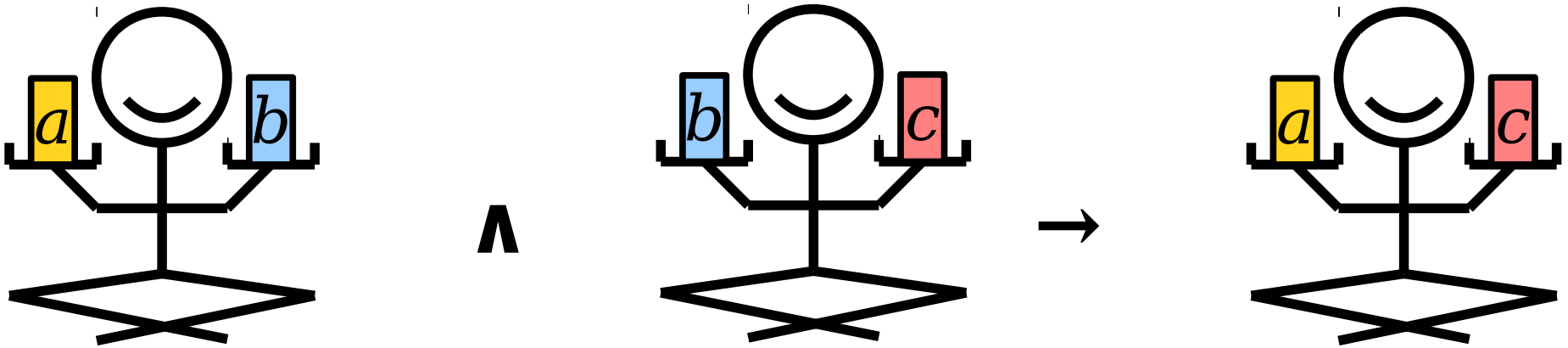

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$$aRa$$

$$aRb \quad \rightarrow \quad bRa$$

$$aRb \quad \wedge \quad bRc \quad \rightarrow \quad aRc$$

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$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

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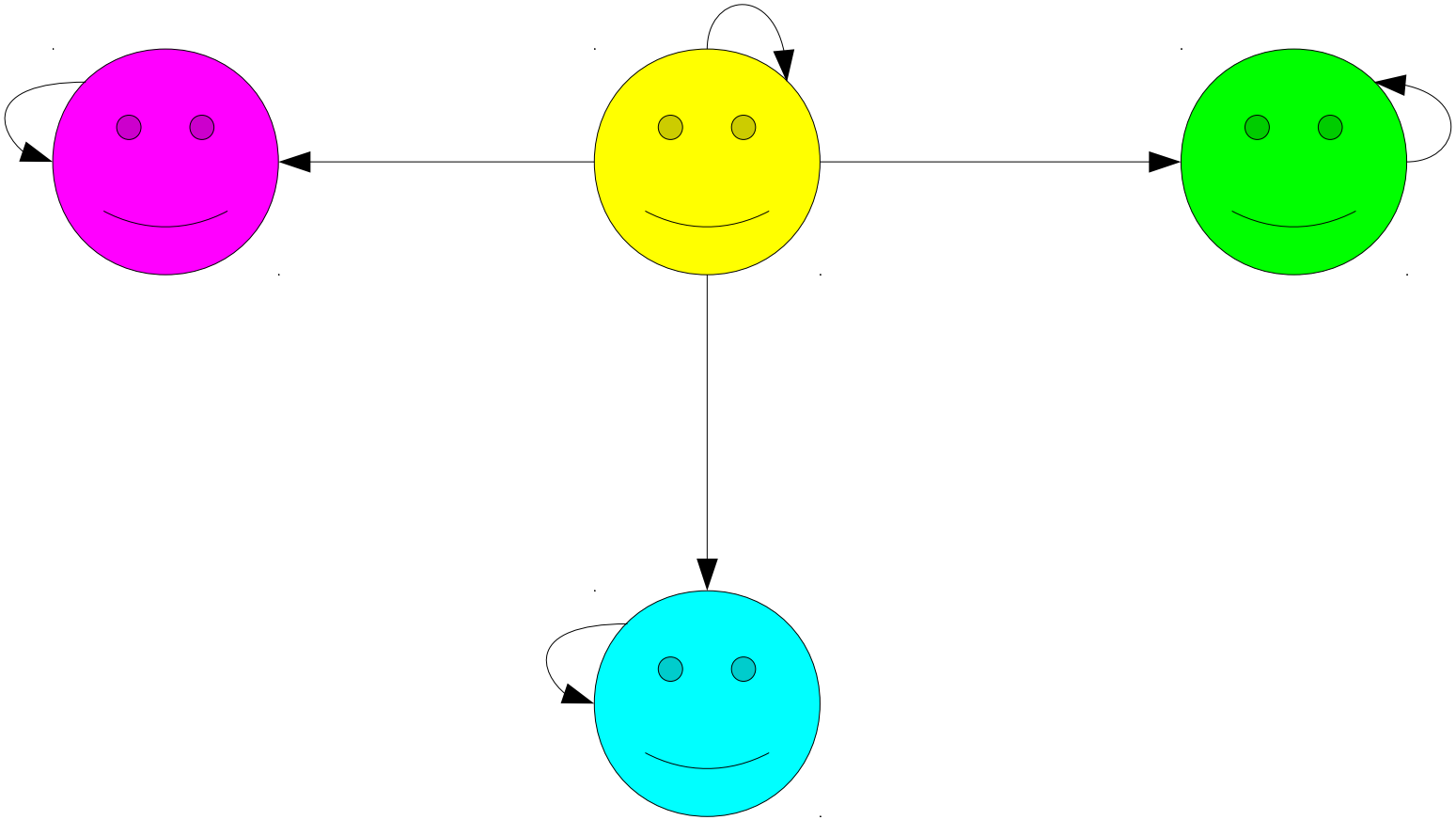
Reflexivity

- In some relations, it happens that *every element in the set relates to itself*.
- Examples:
  - $x = x$ for any x.
  - $A \subseteq A$ for any set A.
  - $x \equiv_k x$ for any x.
- Relations of this sort are called ***reflexive***.
- Formally speaking, a binary relation R over a set A is reflexive if the following first-order statement is true:

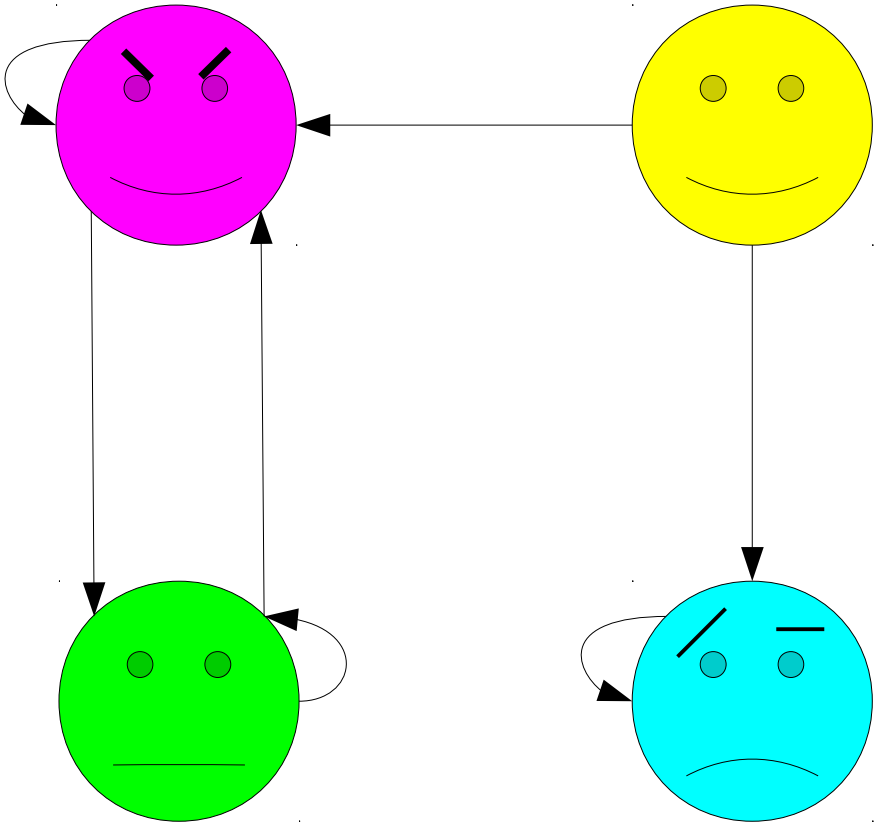
$$\forall a \in A.\ aRa$$

Reflexivity Visualized

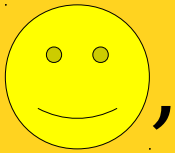
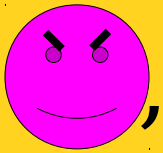
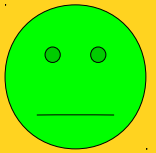


$$\forall a \in A.\ aRa$$

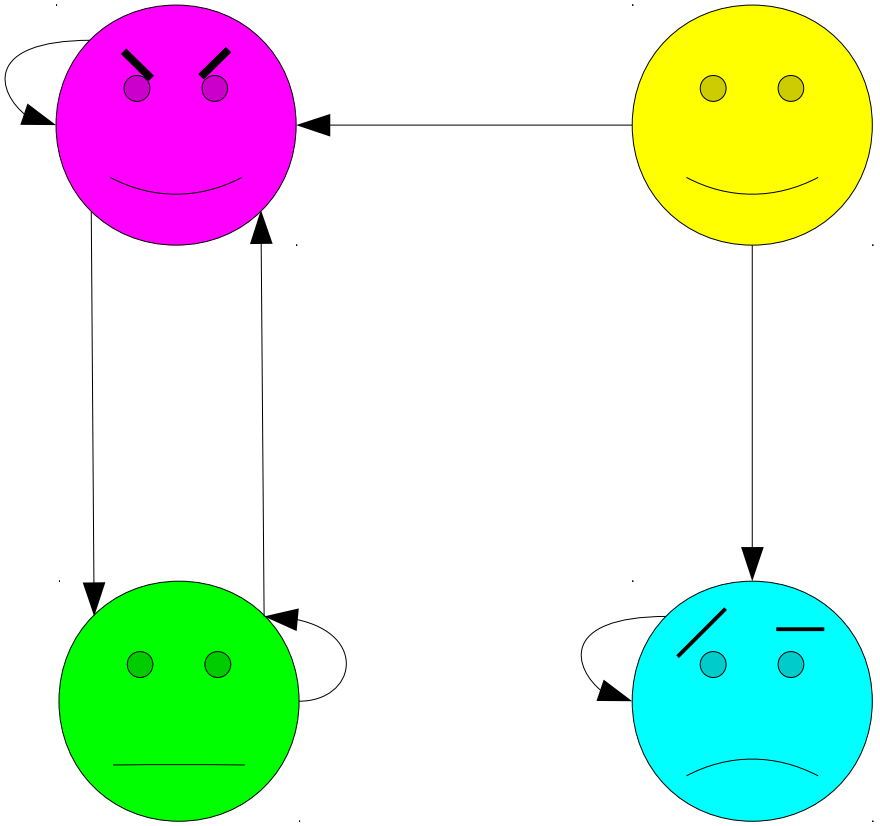
Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **0**, **1**, **2**, **3**, **4**, or **5**.



Let R be the binary relation given by the drawing to the left. How many of the following objects are reflexive?

R,  ,  ,  , 

$$\forall a \in A.\ aRa$$



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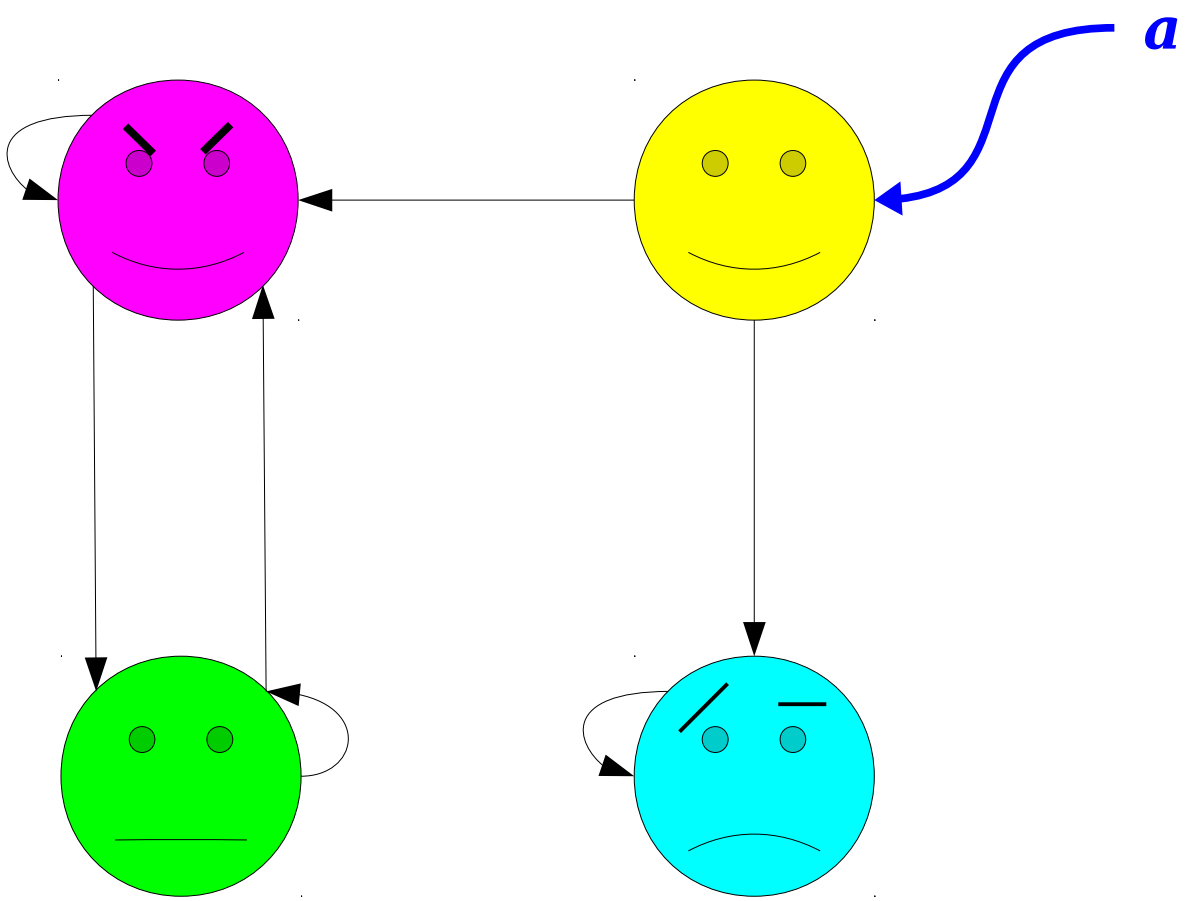

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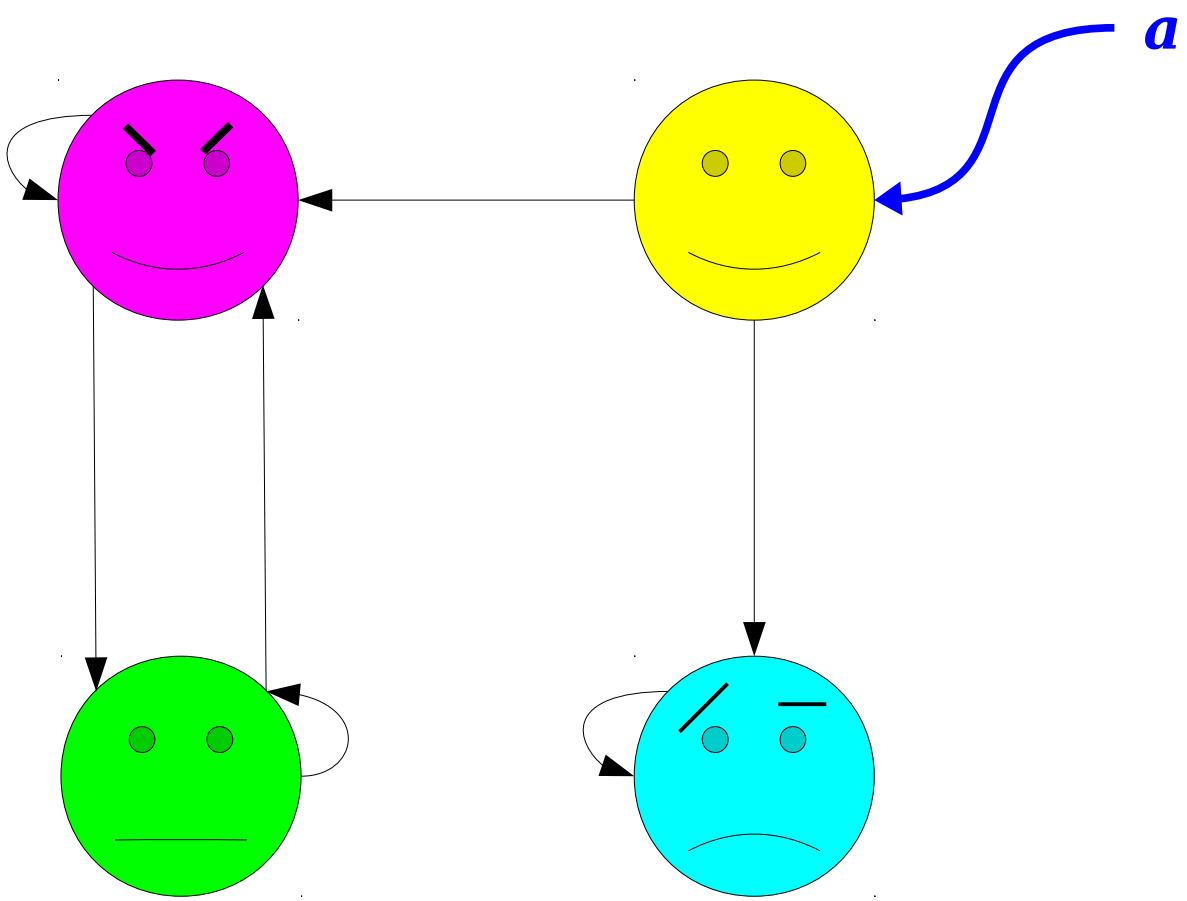

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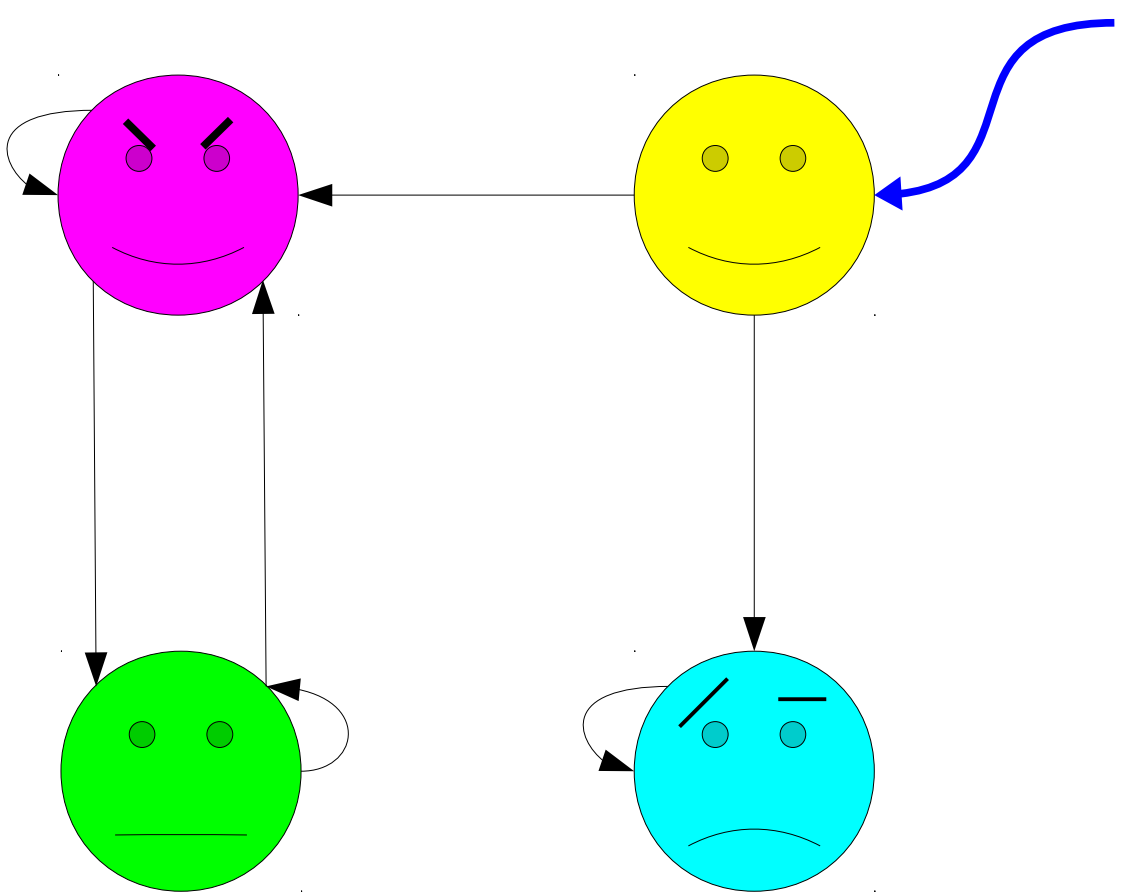
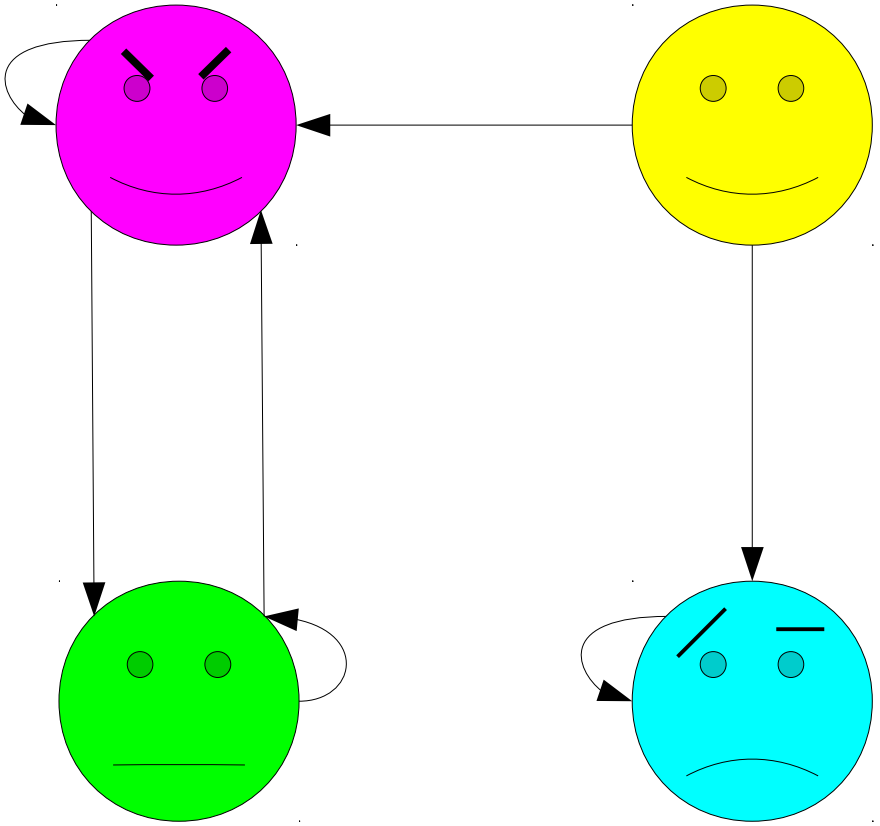
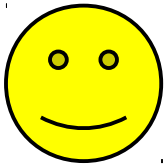
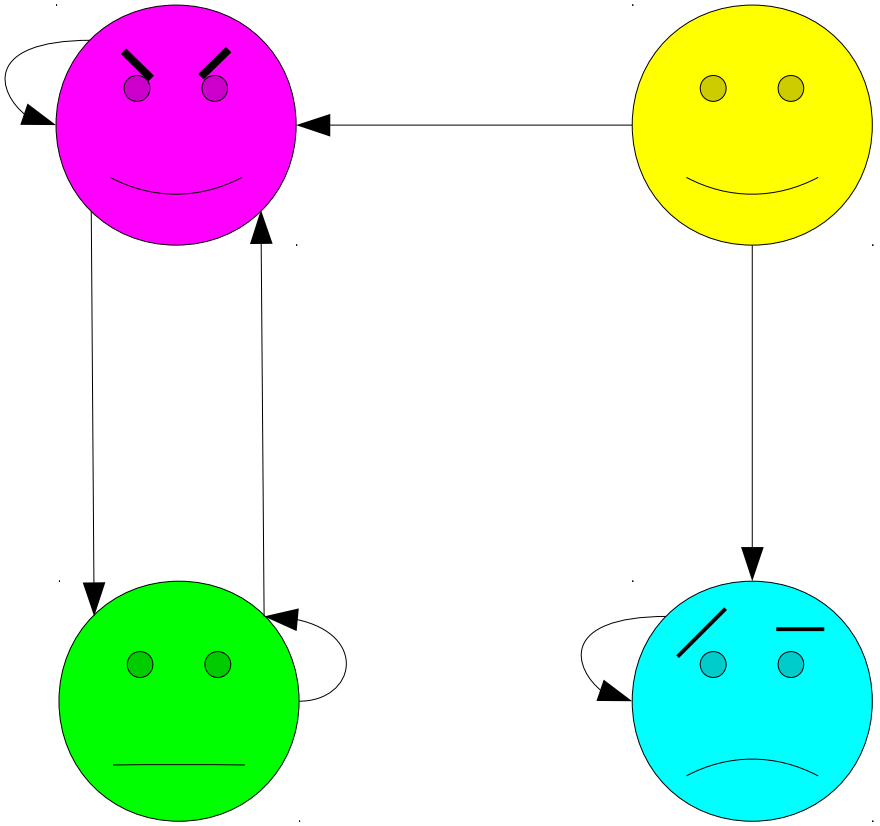


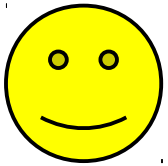
This means that R is not reflexive, since the first-order logic statement given below is not true.

$$\forall a \in A.\ aRa$$

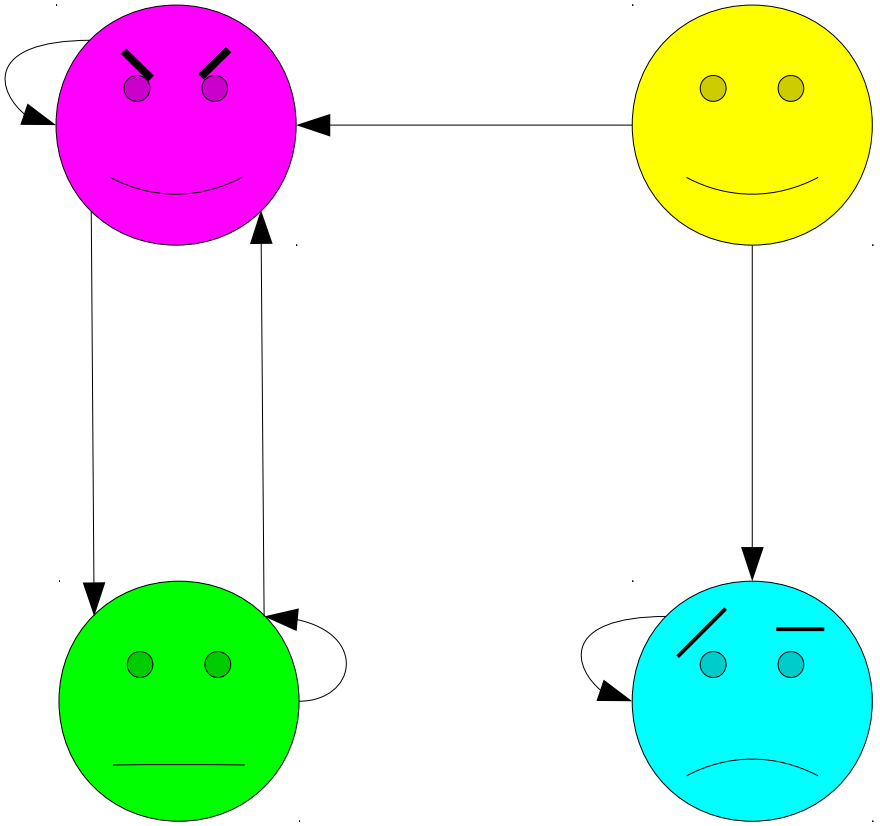


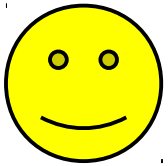
Is  reflexive?



Is  reflexive?

$\forall a \in ??.\ a \; a$



Is  reflexive?

Reflexivity is a property of ***relations***, not ***individual objects***.

Relation this person holds: "Are these two things in the same partition?" for some mystery partition.

$$\forall a \in A.\ aRa$$

$$\forall a \in A.\ \forall b \in A.\ (aRb \to bRa)$$

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$$

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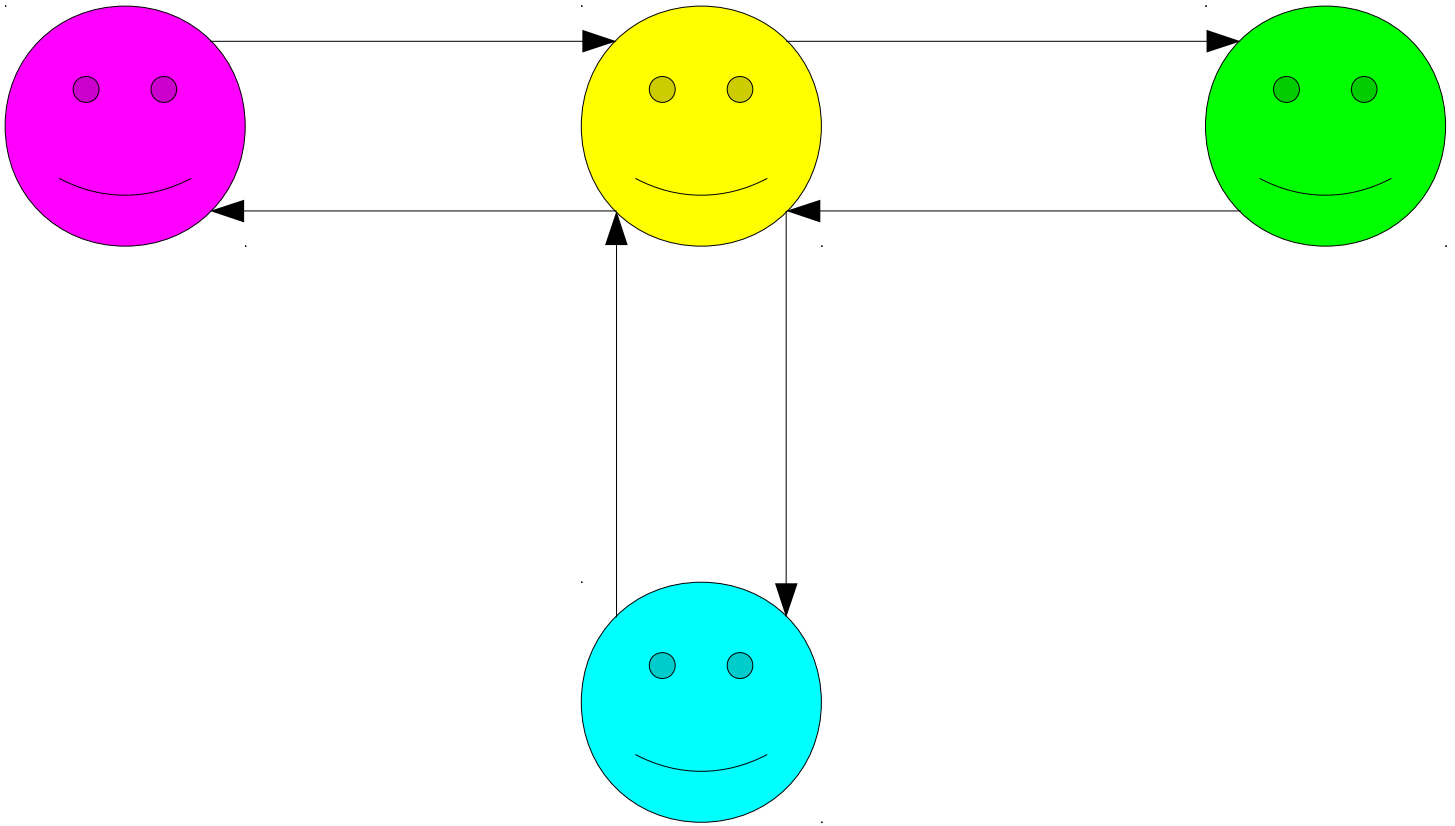
Symmetry

- In some relations, the relative order of the objects doesn't matter. In other words, *if a is related to b, then b is related to a.*
- Examples:
  - If $x = y$, then $y = x$.
  - If $x \equiv_k y$, then $y \equiv_k x$.
- These relations are called ***symmetric***.
- Formally: a binary relation R over a set A is called *symmetric* if the following first-order statement is true about R:

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow bRa)$$

Symmetry Visualized



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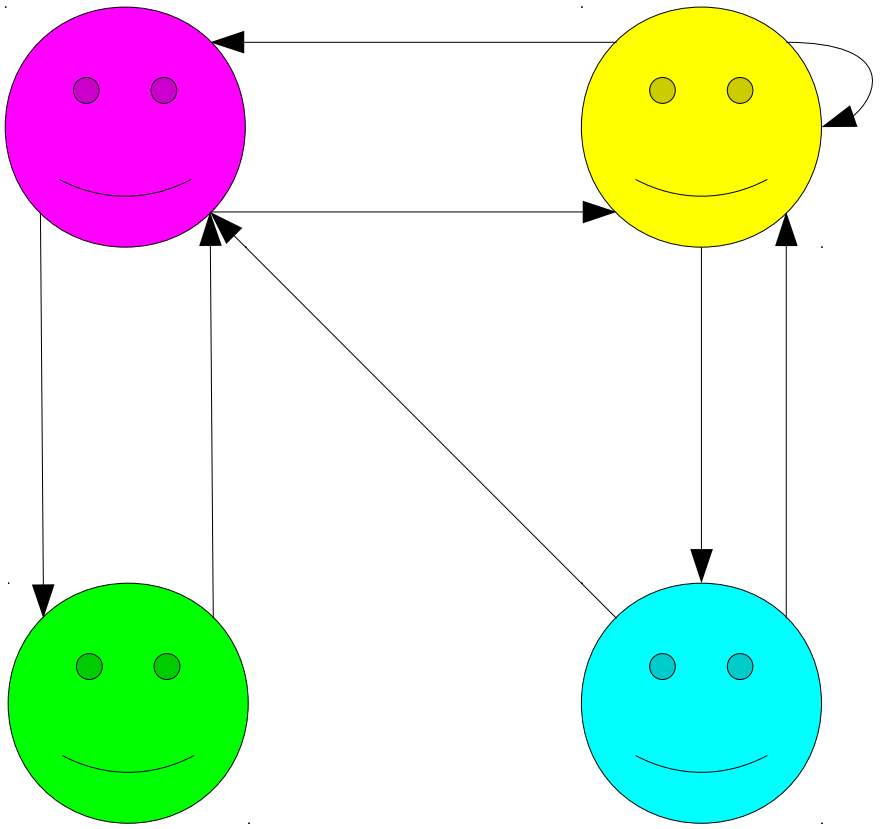
Is This Relation Symmetric?



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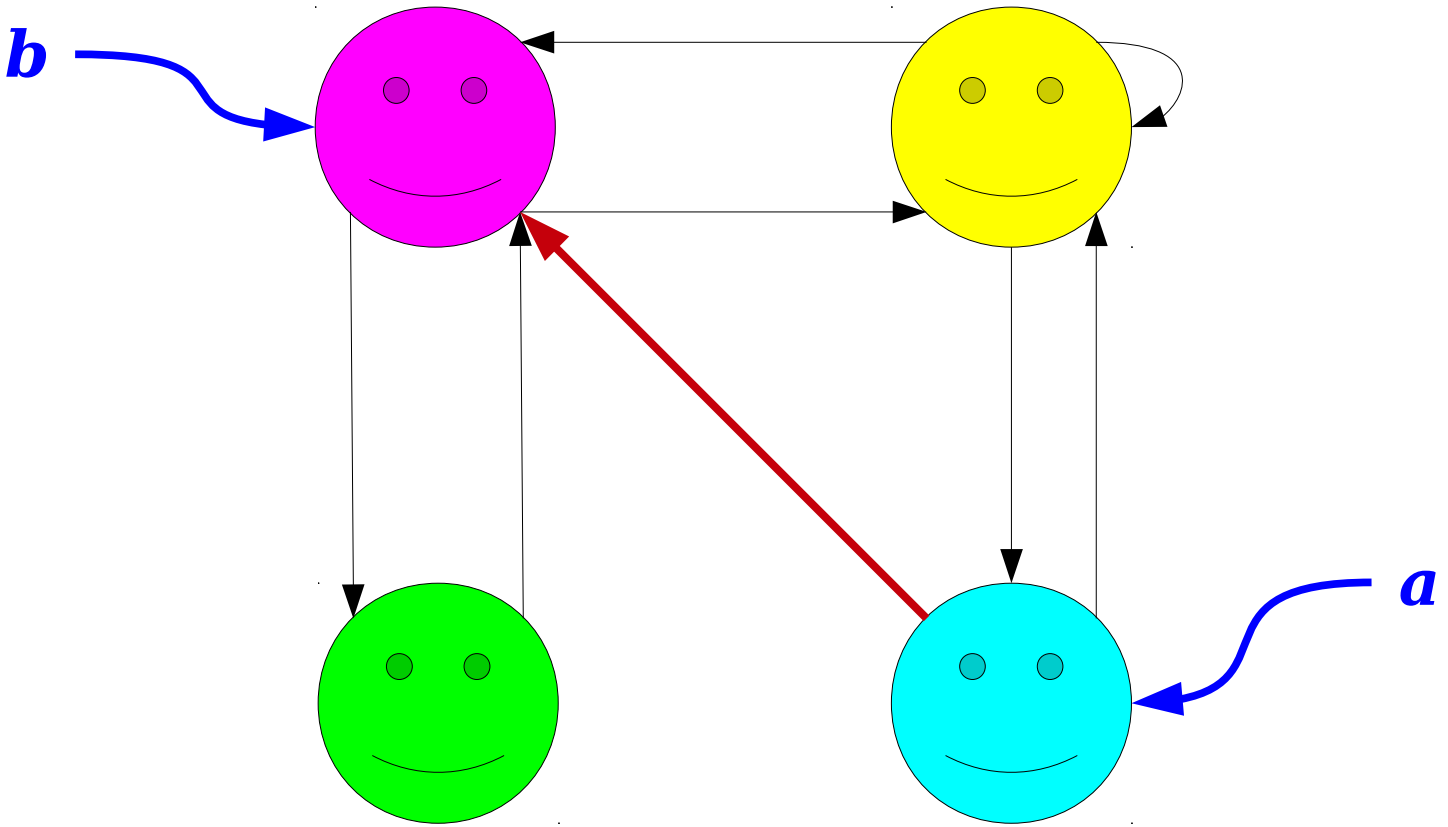

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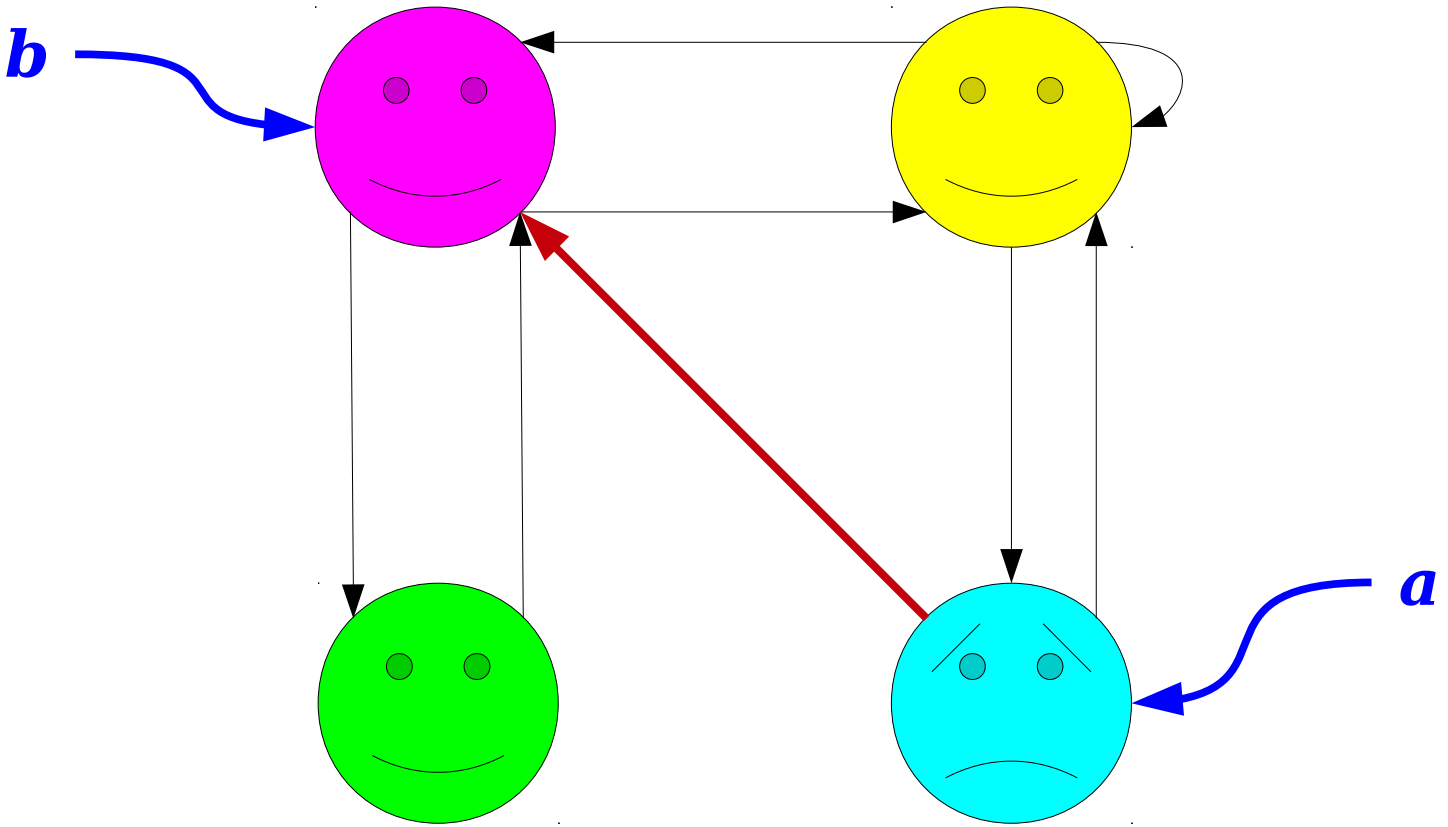

$$\forall a \in A.\, \forall b \in A.\, (aRb \rightarrow bRa)$$

Is this relation symmetric?

Answer at **PollEv.com/cs103** or
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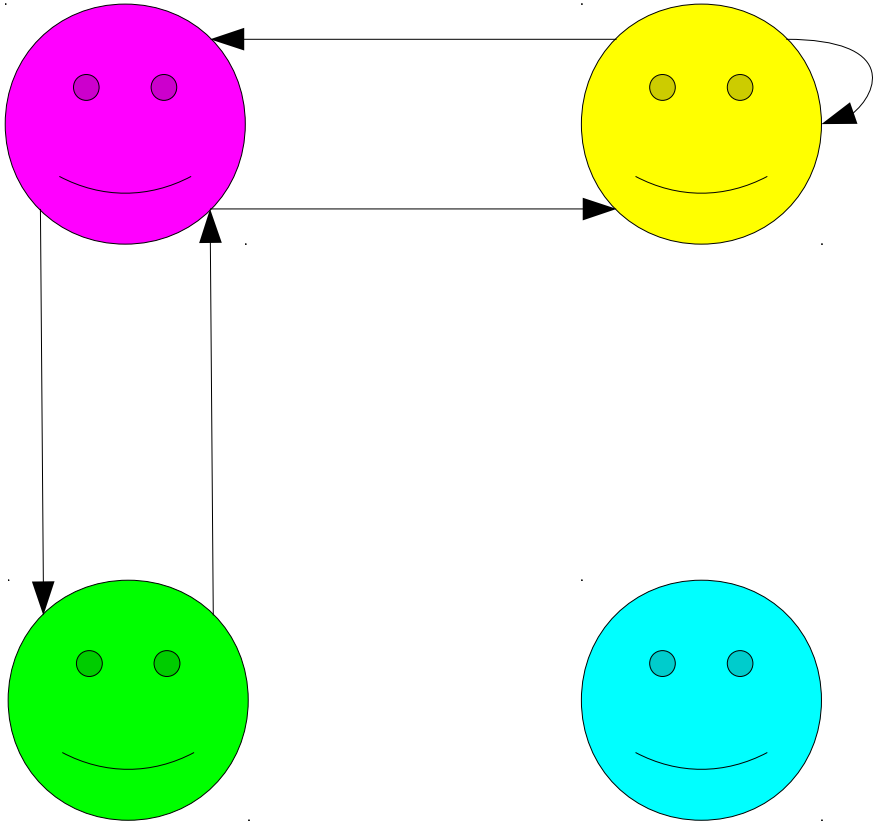

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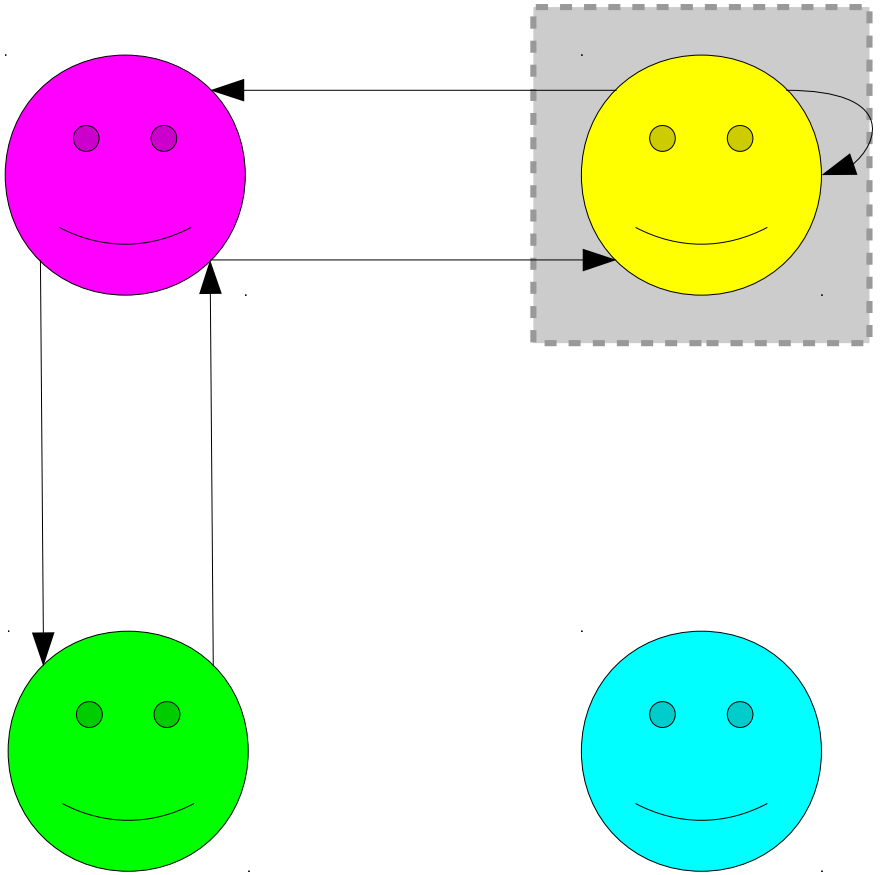
Is This Relation Symmetric?



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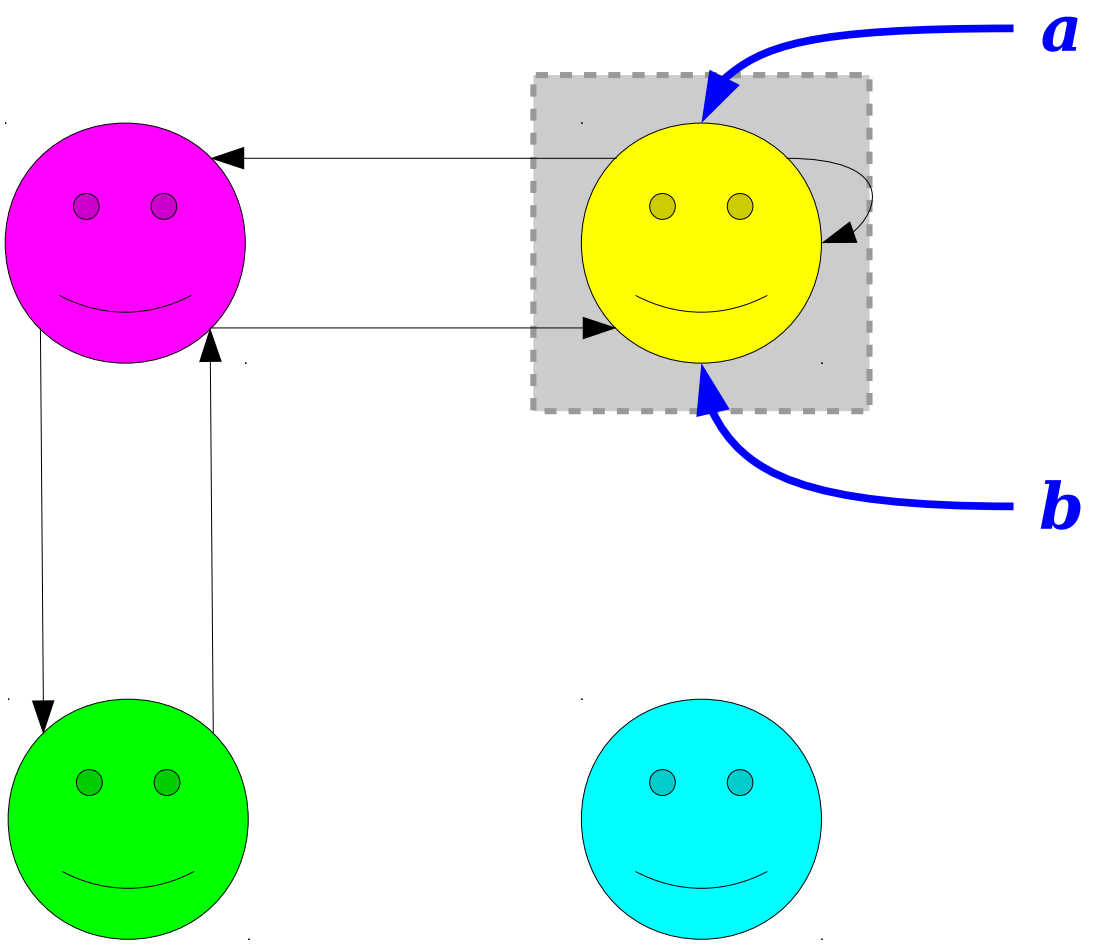

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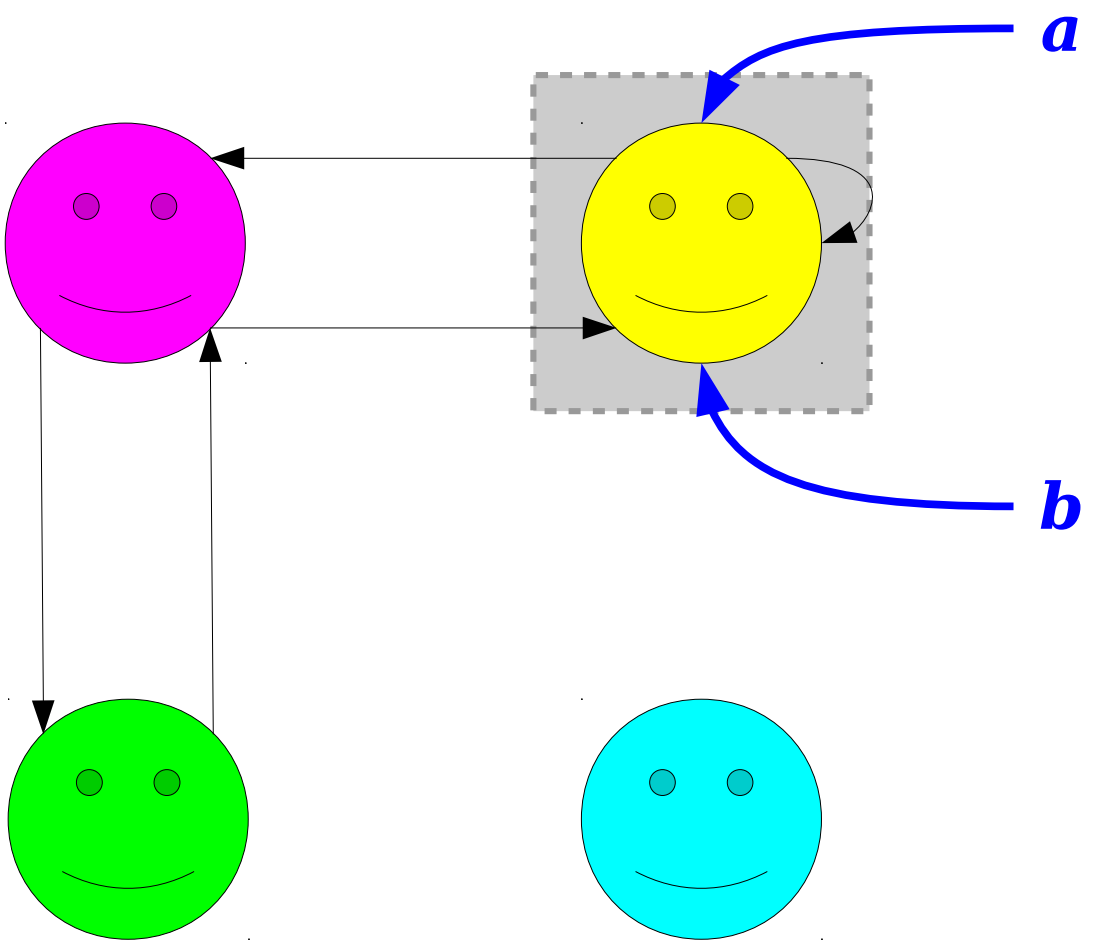

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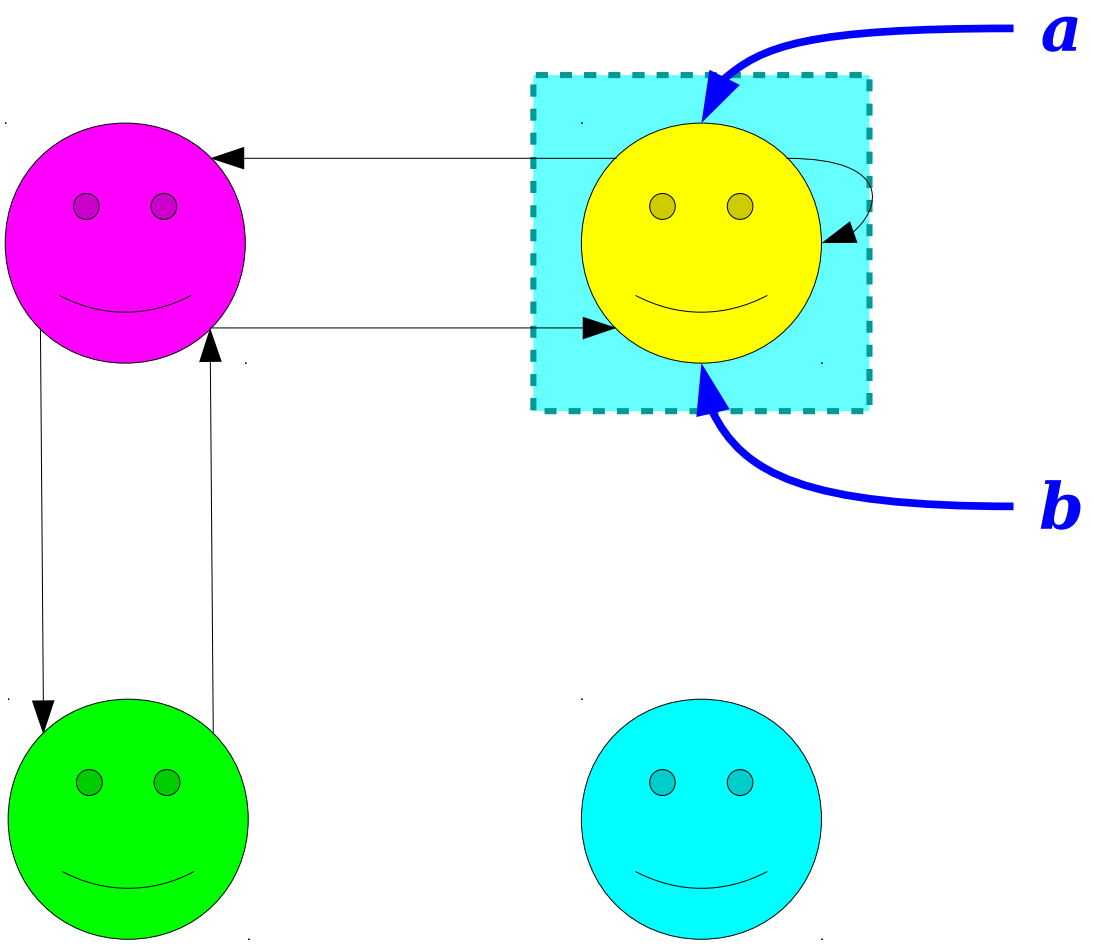

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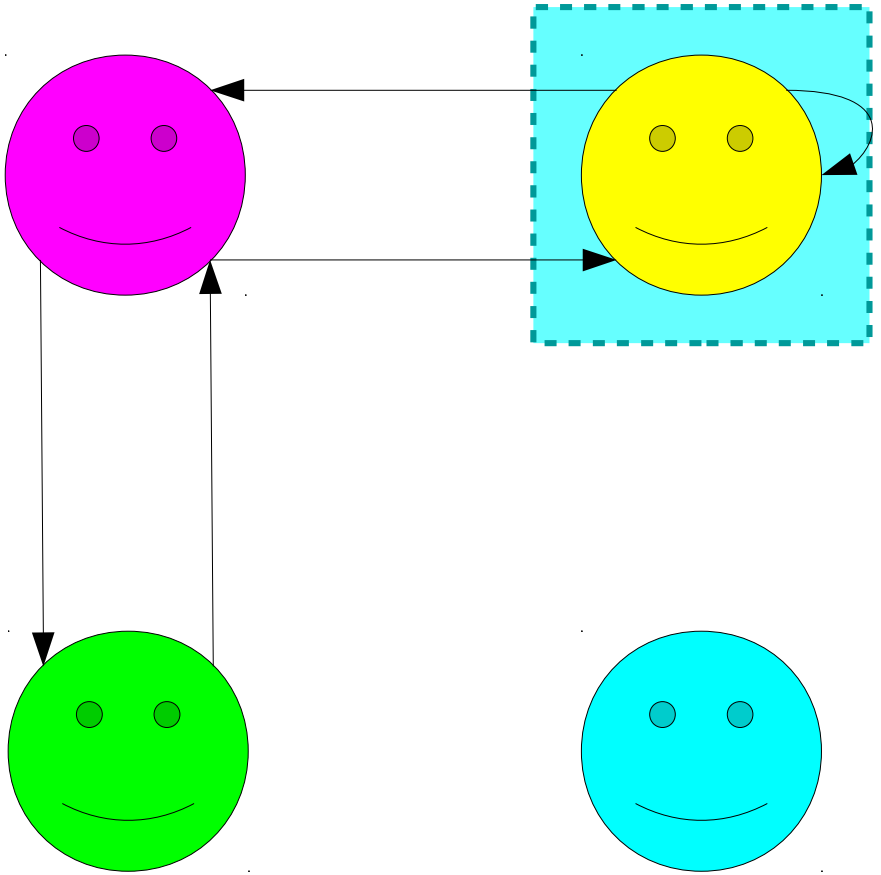

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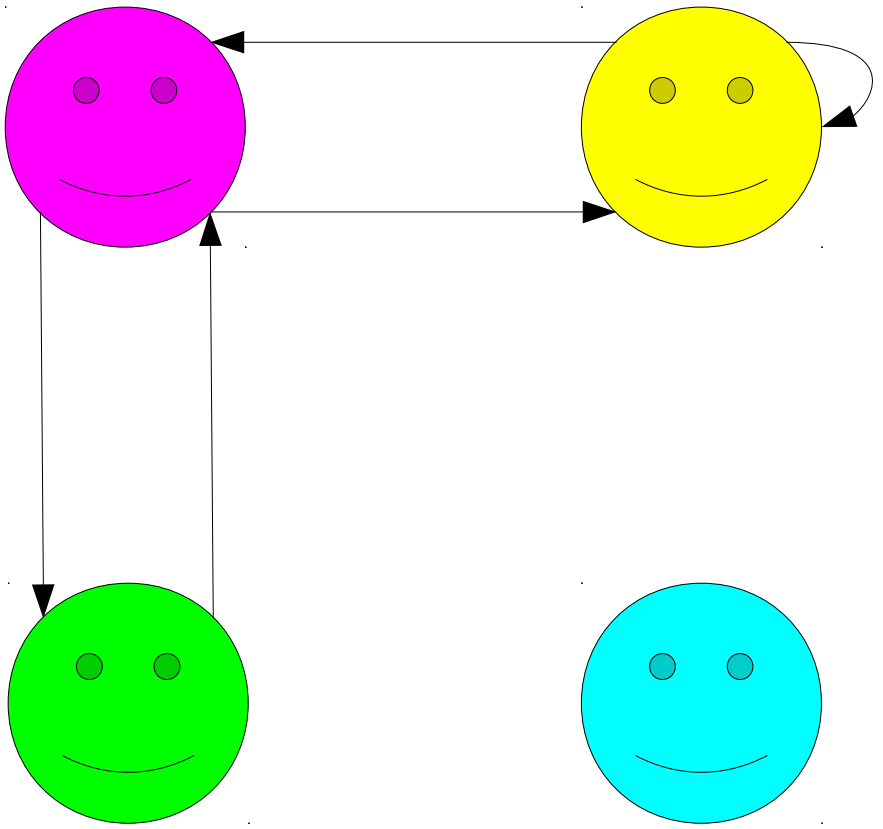

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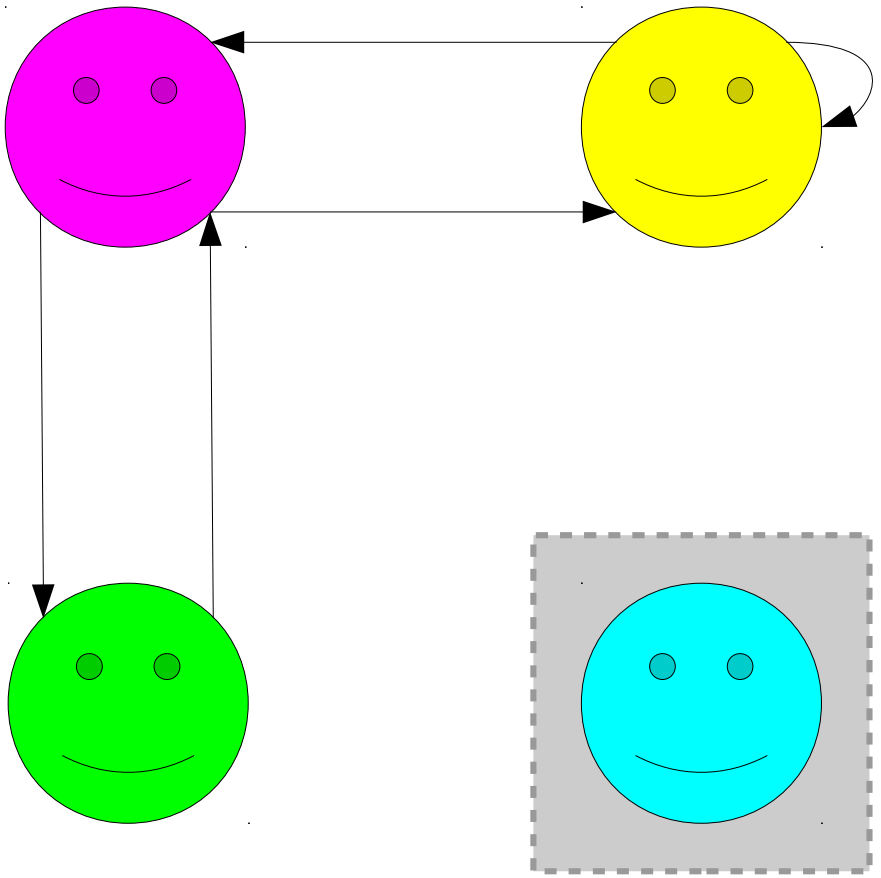

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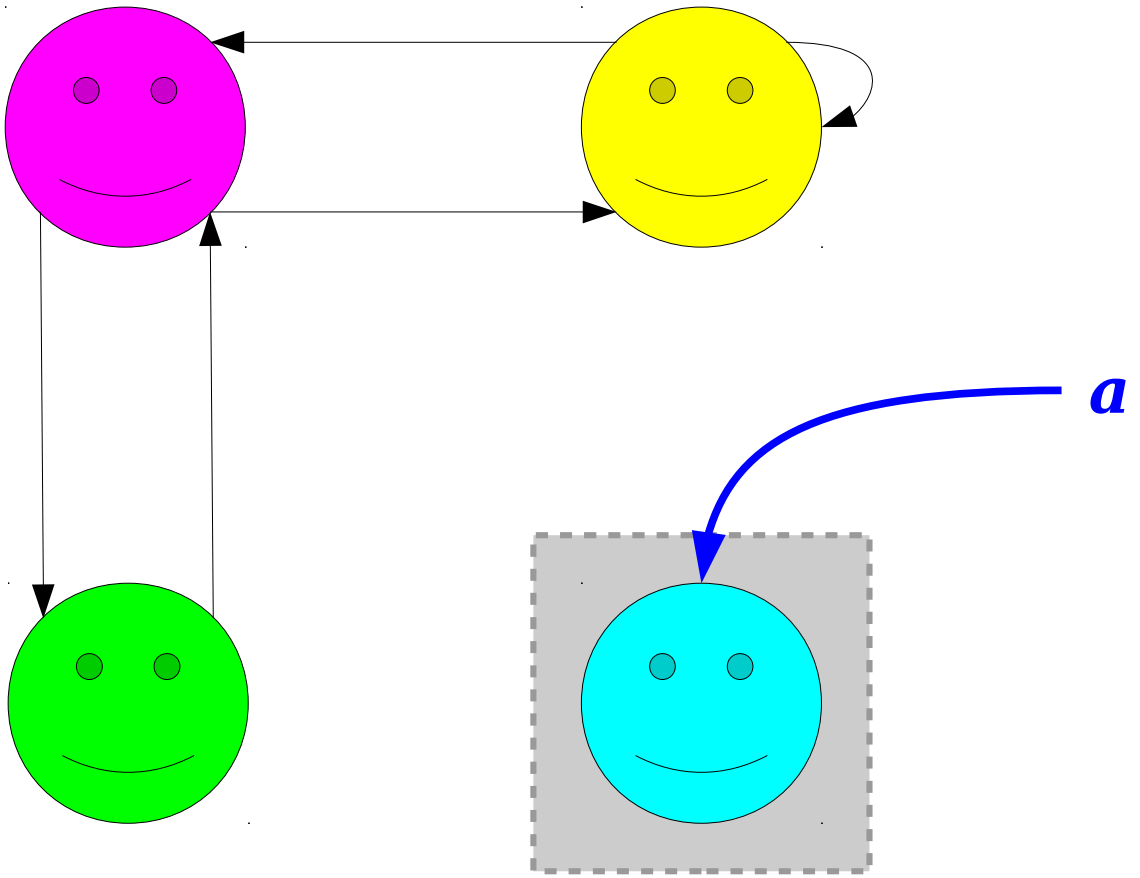

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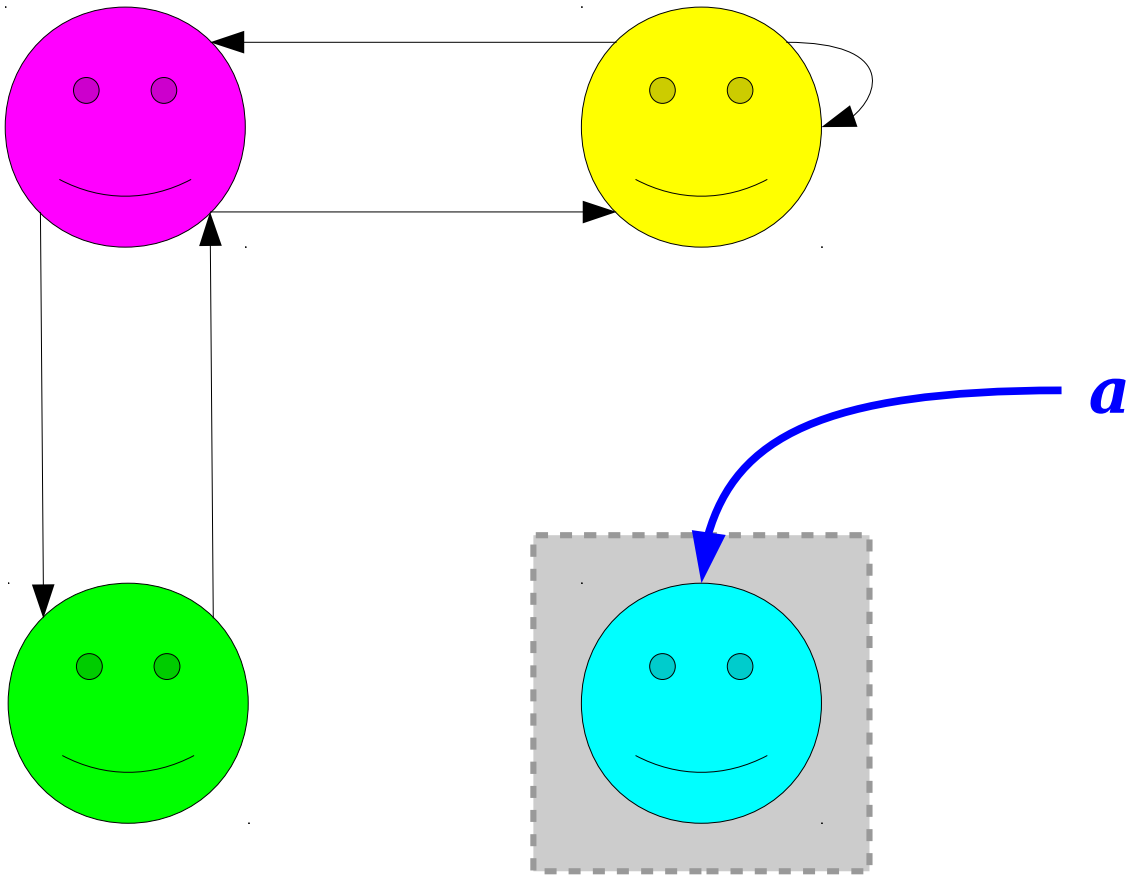

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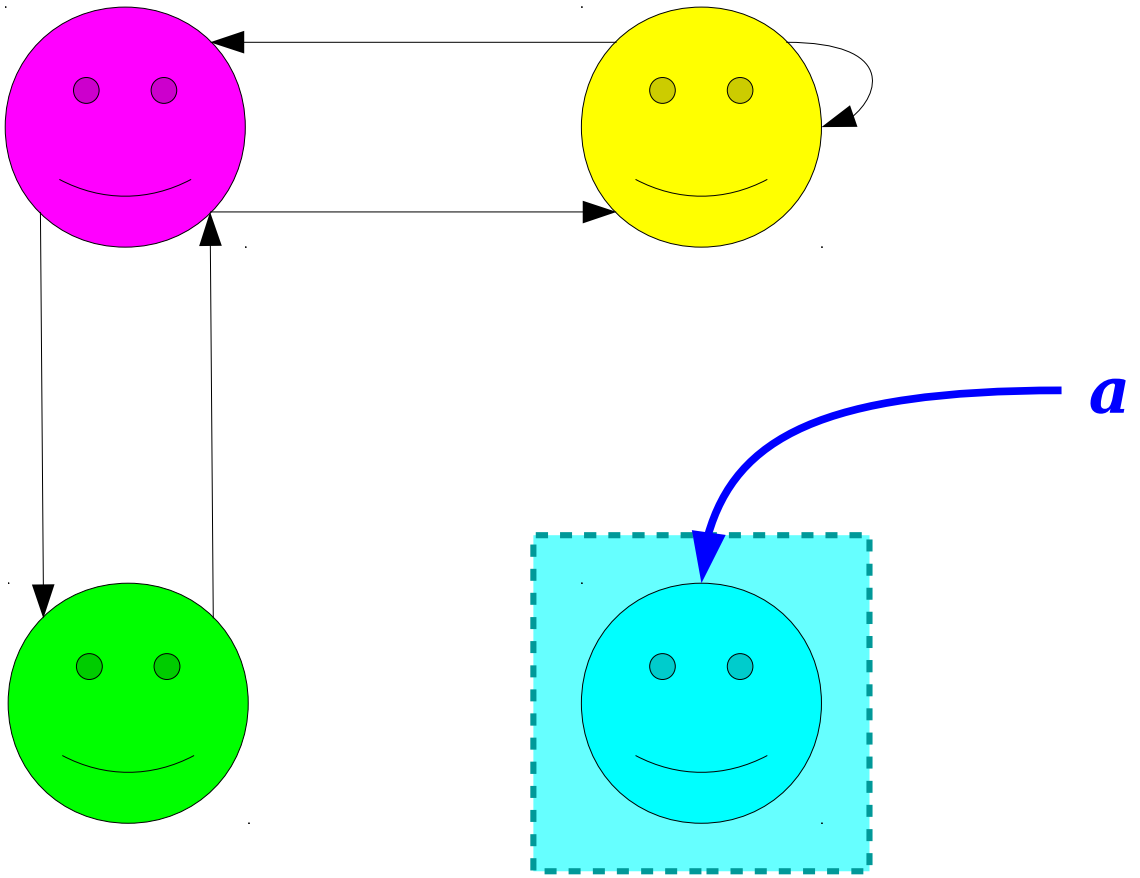

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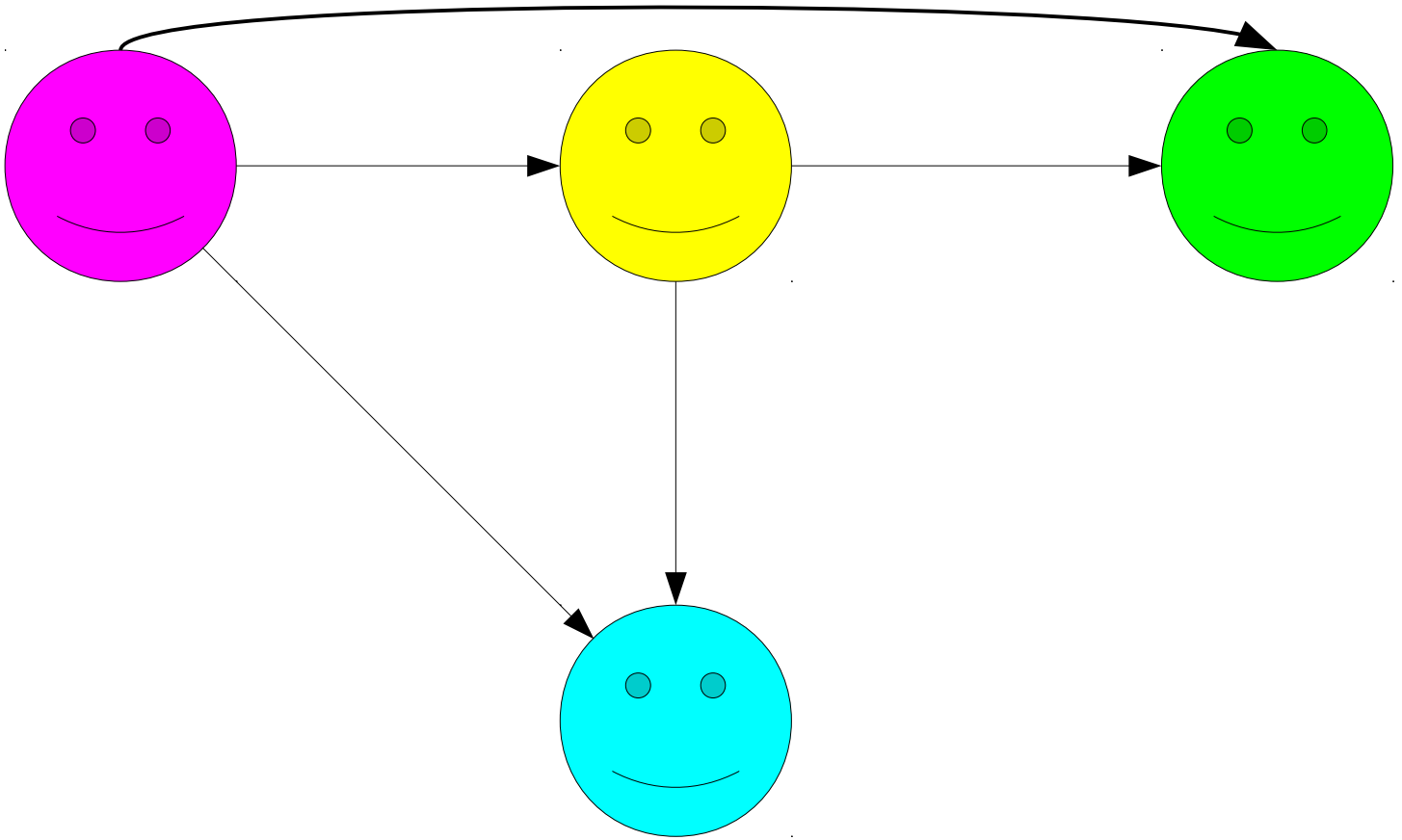
Transitivity

- In some relations, it happens that whenever a is related to b and b is related to c, we know a is related to c
- Examples:
  - If $x = y$ and $y = z$, then $x = z$.
  - If $R \subseteq S$ and $S \subseteq T$, then $R \subseteq T$.
  - If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.
- These relations are called ***transitive***.
- A binary relation R over a set A is called *transitive* if the following first-order statement is true about R:

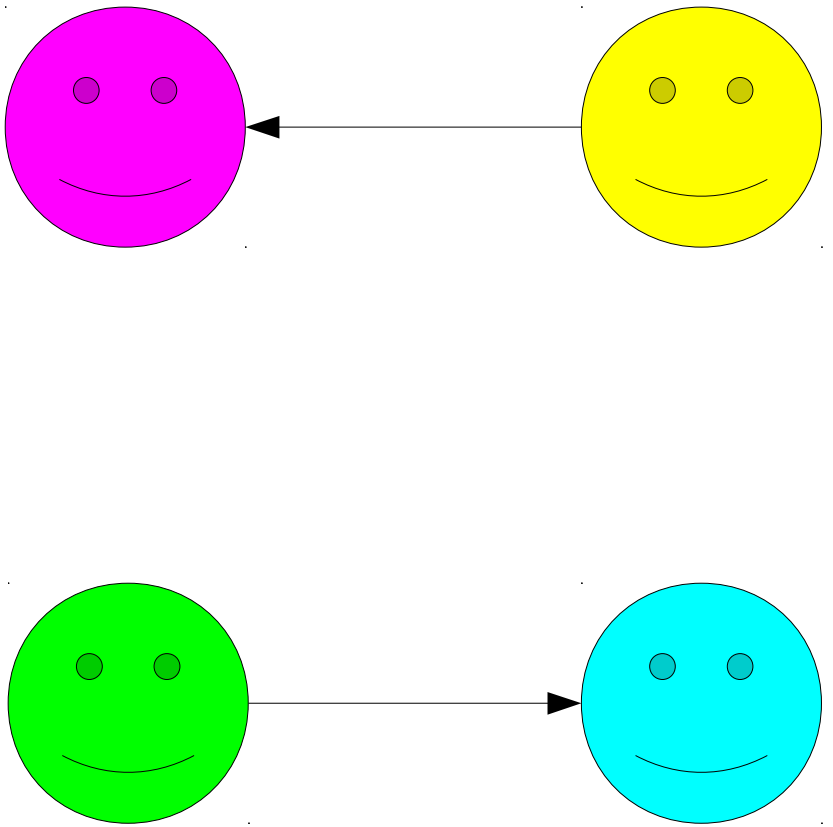
$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \rightarrow aRc)$$

Transitivity Visualized



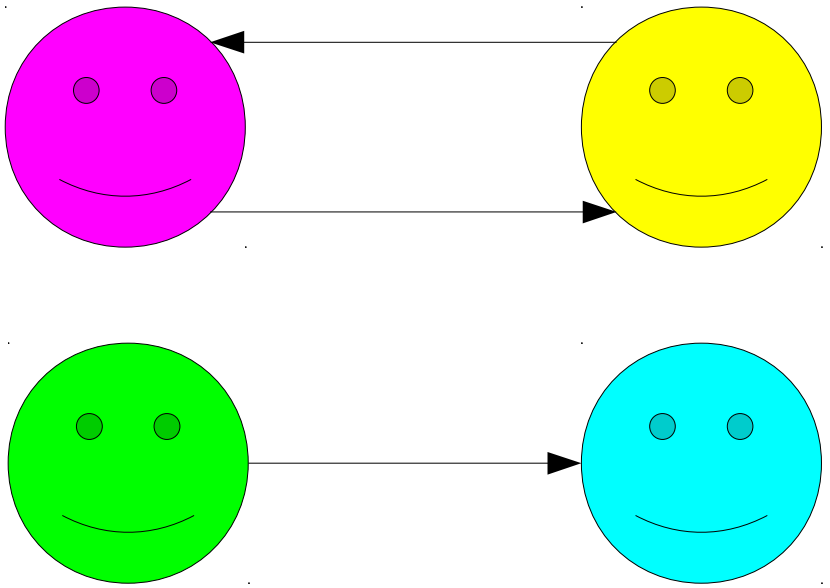
$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \rightarrow aRc)$$

Is This Relation Transitive?



$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \wedge bRc \rightarrow aRc)$$

Is this relation transitive?



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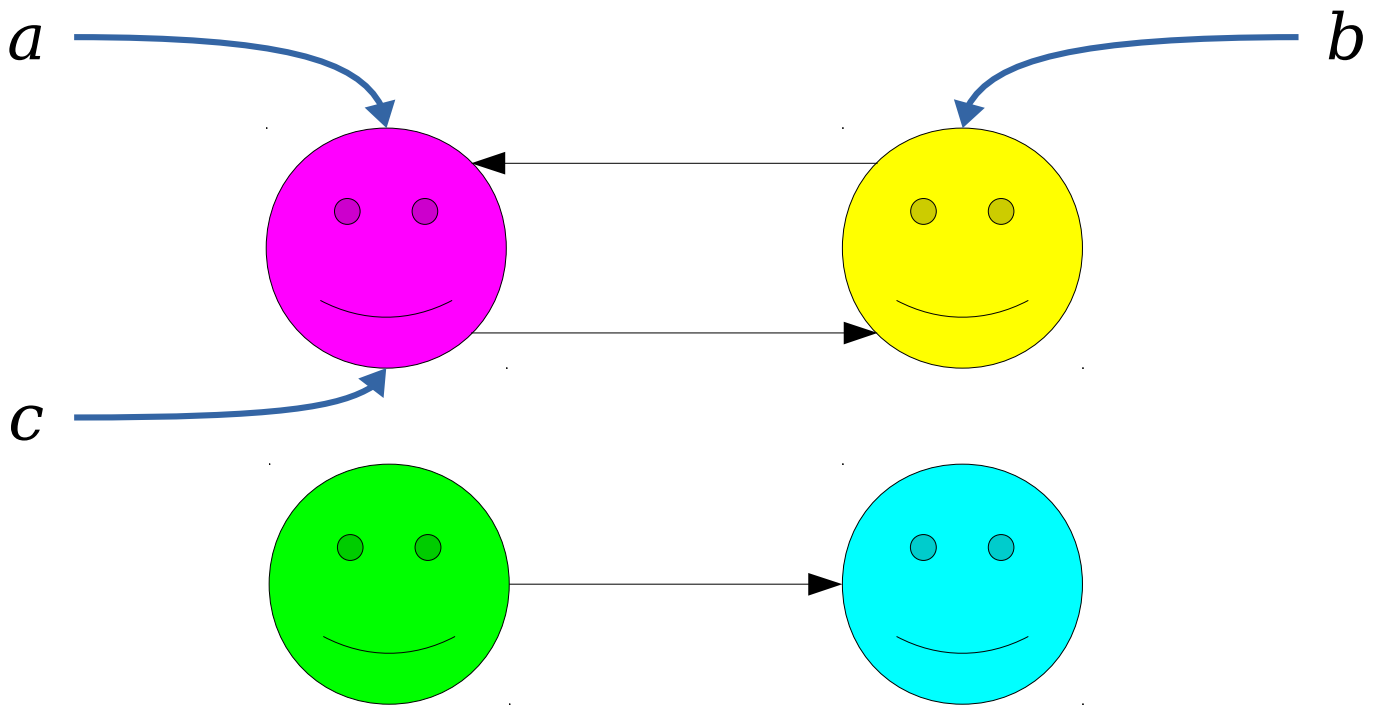
Is This Relation Transitive?



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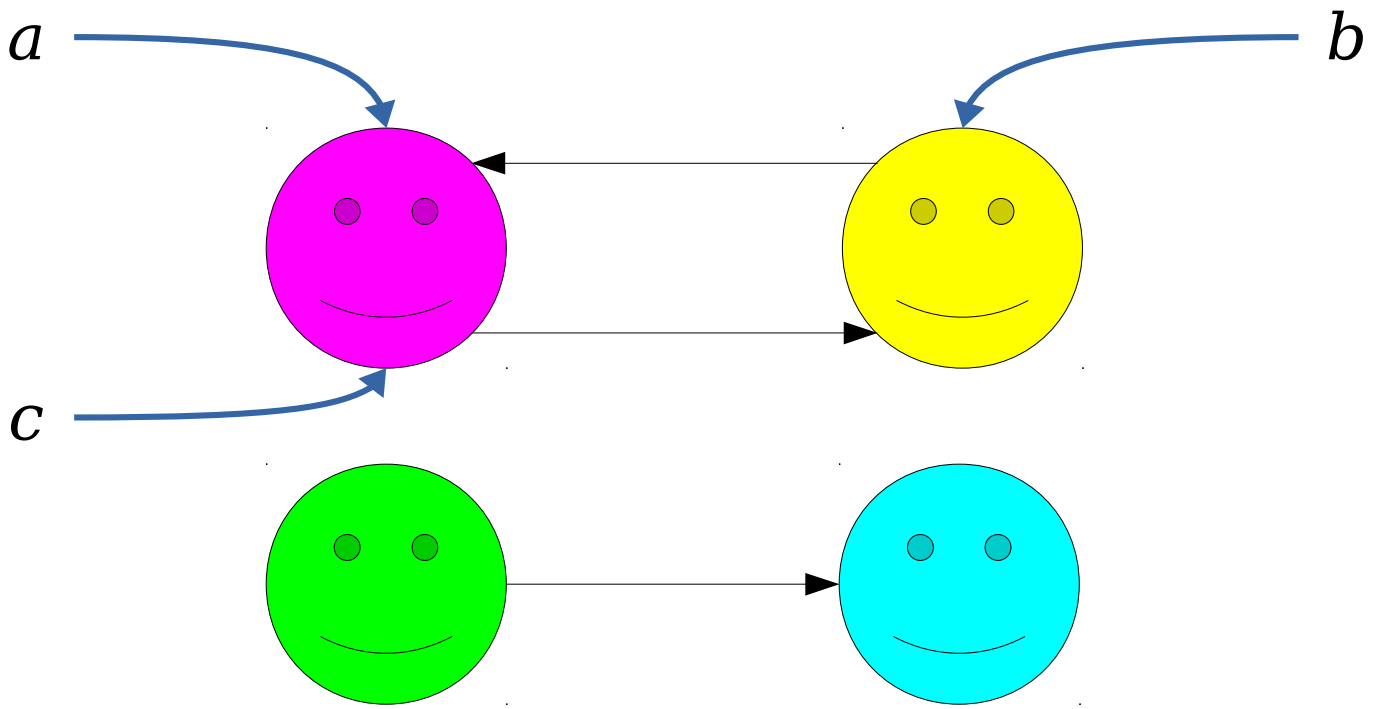

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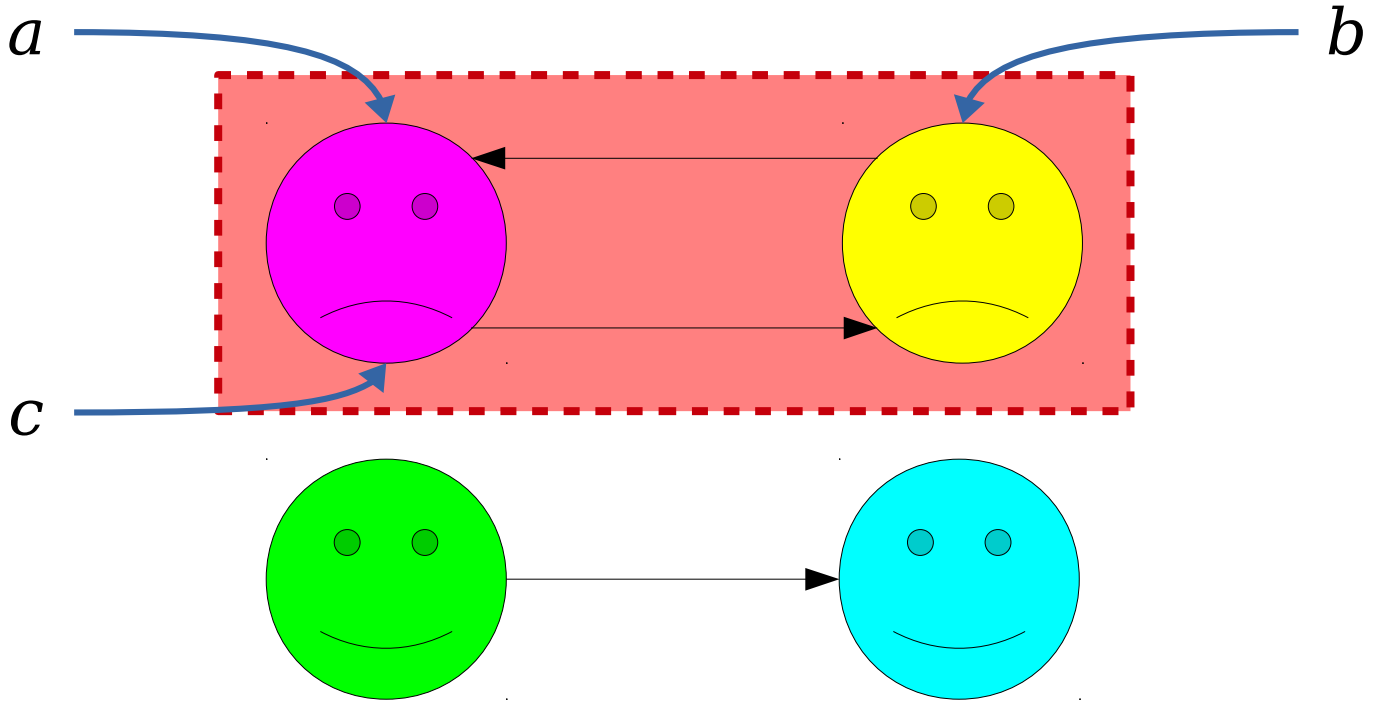

$$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \wedge bRc \rightarrow aRc)$$

Equivalence Relations

- An ***equivalence relation*** is a relation that is reflexive, symmetric and transitive.
- Some examples:
  - $x = y$
  - $x \equiv_k y$
  - x has the same color as y
  - x has the same shape as y.

Equivalence Relations

- Most modern programming languages include some sort of hash table data structure.
  - Java: `HashMap`
  - C++: `std::unordered_map`
  - Python: `dict`
- If you insert a key/value pair and then try to look up a key, the implementation has to be able to tell whether two keys are equal.
- Although each language has a different mechanism for specifying this, many languages describe them in similar ways...

Equivalence Relations

"The `equals` method implements an equivalence relation on non-null object references:

- It is *reflexive*: for any non-null reference value `x`, `x.equals(x)` should return true.
- It is *symmetric*: for any non-null reference values `x` and `y`, `x.equals(y)` should return true if and only if `y.equals(x)` returns true.
- It is *transitive*: for any non-null reference values `x`, `y`, and `z`, if `x.equals(y)` returns `true` and `y.equals(z)` returns true, then `x.equals(z)` should return `true`."

Java 8 Documentation

Equivalence Relations

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Java 8 Documentation

Equivalence Relations

"Each unordered associative container is parameterized by `Key`, by a function object type `Hash` that meets the Hash requirements (17.6.3.4) and acts as a hash function for argument values of type `Key`, and by a binary predicate `Pred` that induces an equivalence relation on values of type `Key`. Additionally, `unordered_map` and `unordered_multimap` associate an arbitrary mapped type `T` with the `Key`."

C++14 ISO Spec, §23.2.5/3

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Equivalence Relation Proofs

- Let's suppose you've found a binary relation R over a set A and want to prove that it's an equivalence relation.
- How exactly would you go about doing this?